Instituto Superior Técnico
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Quantum Probabilistic Graphical Models for Decision and Cognition

by **Catarina Pinto Moreira**

Structured Machine Learning Course 2013-2014

Outline

- Motivation Example
- Classical Probability vs Quantum Probability
- Violations of Probability Theory (related Work)
- Classical Bayesian Networks
- Quantum Bayesian Networks

Motivation

The Sure Thing Principle (Savage, 1954):

*If one chooses **action A** over **B** under state of the **world X** and if one also chooses **action A** over **B** in the state of the **world ¬X**, then one should always choose **action A** over **B** even if the state of the **world** is **unknown**.*

L. Savage (1954), *The Foundations of Statistics*. John Wiley & Sons Inc.

Motivation – The Two Stage Gambling Game

Motivation – The Two Stage Gambling Game

Participants were asked to **play a gambling game** that has an equal chance of **winning** $200 or **loosing** $100. **Three** conditions were verified:

- **Informed** that they **won** the 1st gamble;
- **Informed** that they **lost** the 1st gamble;
- **Did not know** if they won or lost the 1st gamble;

A Tversky and E Shafir, E. (1992), 'The disjunction effect in choice under uncertainty', Journal of Psychological Science 3, 305–309

Motivation – The Two Stage Gambling Game

Experimental results:

- Participants who **knew they had won**, decided to **PLAY** again;

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Motivation – The Two Stage Gambling Game

Experimental results:

- Participants who **knew they had won**, decided to **PLAY** again;
- Participants who **knew they had lost**, decided to **PLAY** again;

A Tversky and E Shafir, E. (1992), 'The disjunction effect in choice under uncertainty', Journal of Psychological Science 3, 305–309

Motivation – The Two Stage Gambling Game

The Sure Thing Principle:

State of the world "1st gamble = won"	State of the world "1st gamble = lose"	State of the world "1st gamble = ?"
Action Chosen: Play	Action Chosen: Play	Action Chosen: ? (should be Play)

A Tversky and E Shafir (1992), '*The disjunction effect in choice under uncertainty*', Journal of Psychological Science 3, 305–309

Motivation – The Two Stage Gambling Game

Experimental results:

- Participants who **knew they had won**, decided to **PLAY** again;
- Participants who **knew they had lost**, decided to **PLAY** again;
- Participants who **did not know anything**, decided to **NOT PLAY** again;

A Tversky and E Shafir, E. (1992), 'The disjunction effect in choice under uncertainty', Journal of Psychological Science 3, 305–309

Motivation – The Two Stage Gambling Game

Experimental results:

- Participants who **knew they had won**, decided to **PL**
- Pa ed to **PL**
- Pa cided to **NO** again,

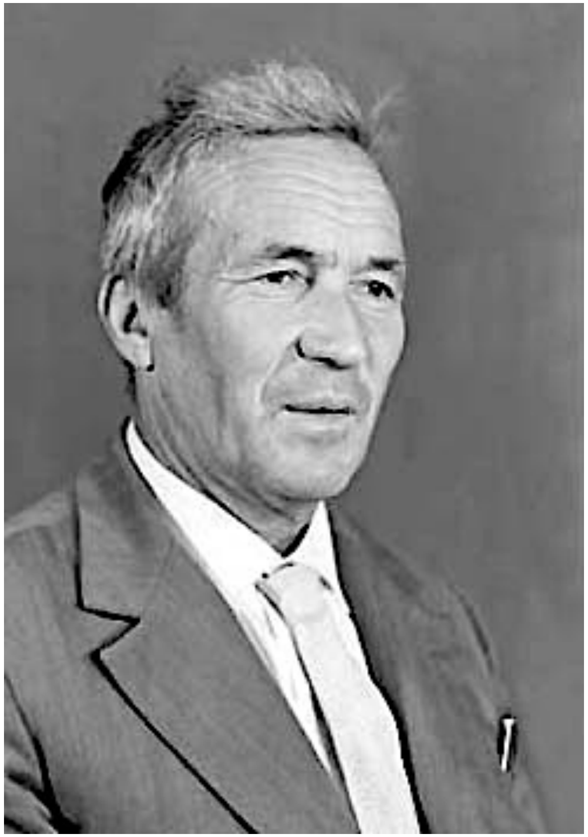
VIOLATES THE SURE THING PRINCIPLE!

A Tversky and E Shafir, E. (1992), 'The disjunction effect in choice under uncertainty', Journal of Psychological Science 3, 305–309

Probability Theory

Two Probability Theories

CLASSICAL PROBABILITY



Andrey Kolmogorov

QUANTUM PROBABILITY

John von Neumann

Classical vs Quantum Probability

Classical vs Quantum Probability

Suppose you are a juror trying to judge whether a defendant is **Guilty** or **Innocent**.

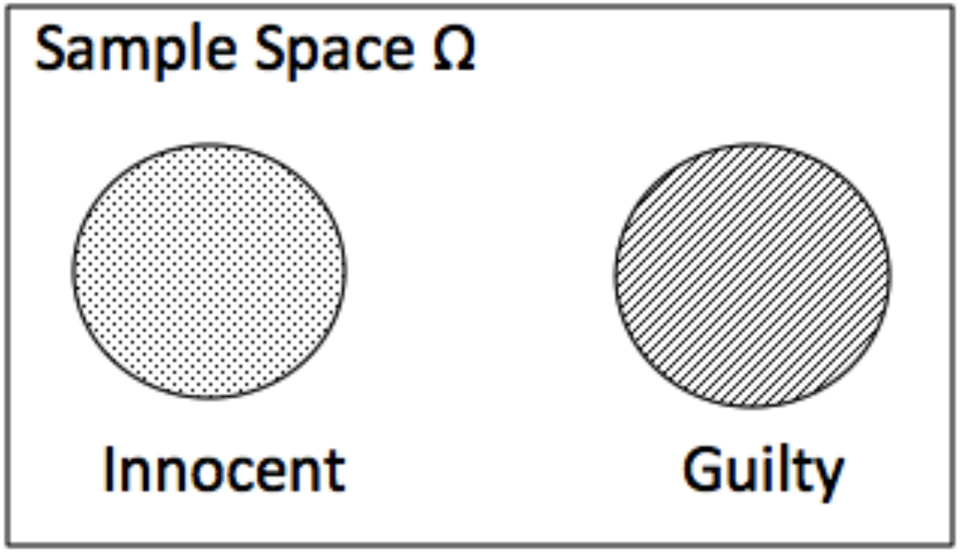
What are the differences between **classical** and **quantum** probabilities?

Classical vs Quantum: Space

CLASSICAL

- Events are contained in a **sample space**, Ω. Corresponds to the set of all possible outcomes.

Ω = { Guilty, Innocent}

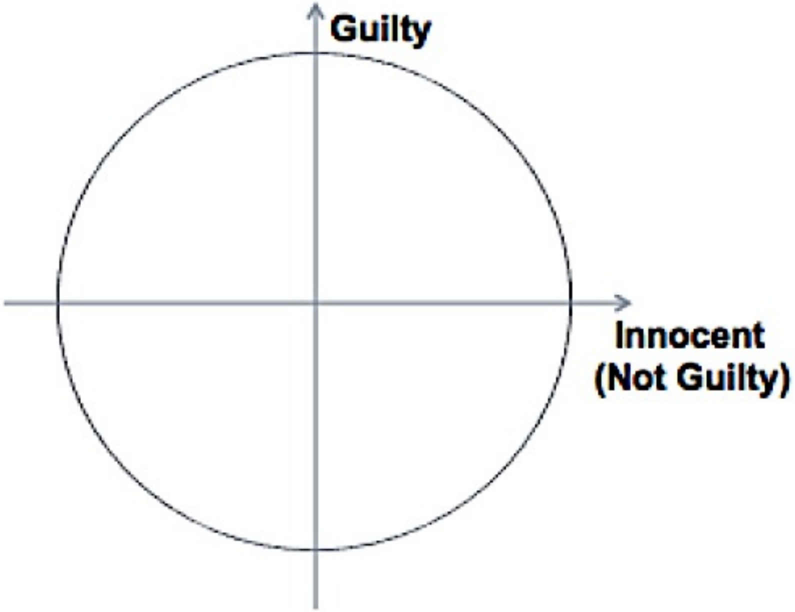


Classical vs Quantum: Space

QUANTUM

- Events are contained in a **Hilbert Space**, *H*.
- Events are **spanned** by a set of **orthonormal** basis vectors, representing all possible outcomes

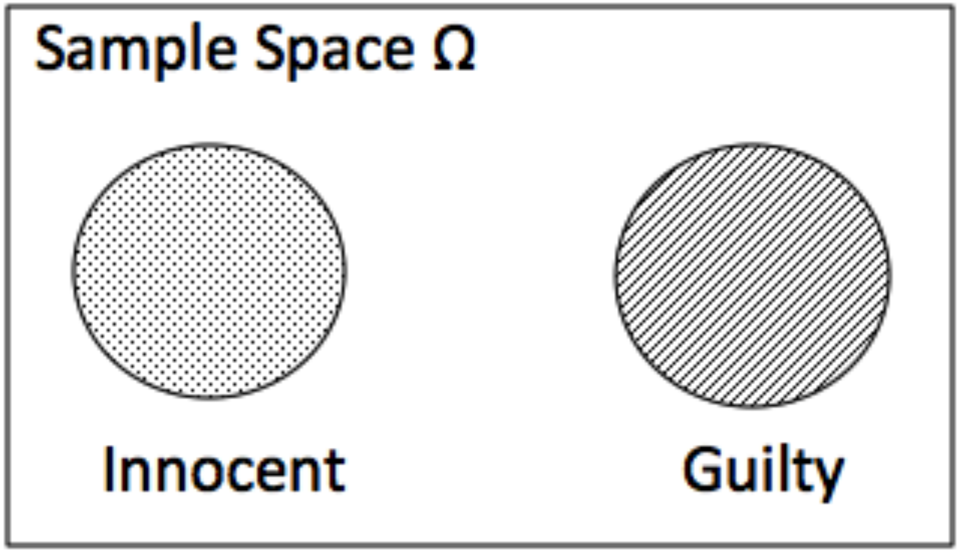
$$H = \{|Guilty\rangle, |Innocent\rangle\}$$



Classical vs Quantum: Events

CLASSICAL

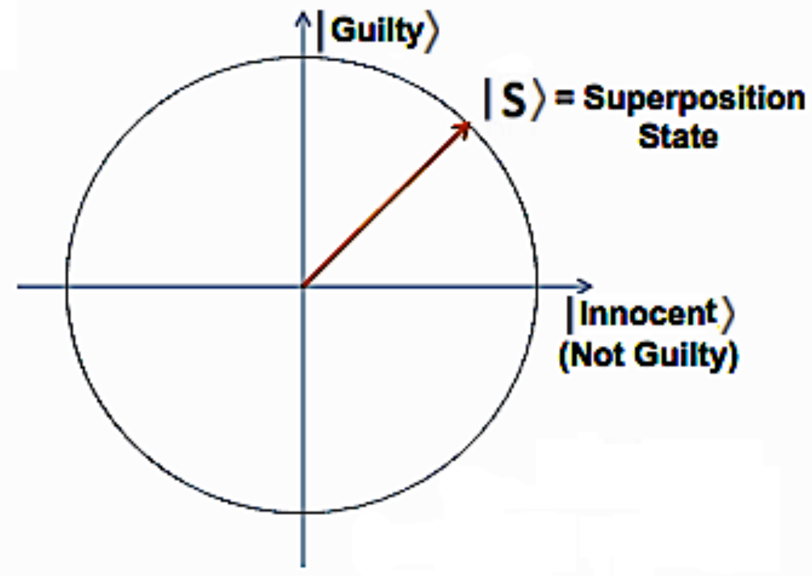
- Can be defined by a set of **outcomes** to which a probability is assigned.
- Can be mutually exclusive and **obey set theory**;
- Operations defined:
  - Intersection
  - Union
  - Distribution



Classical vs Quantum: Events

QUANTUM

- Events correspond to subspaces spanned by a set of basis vectors
- Are defined through a **superposition** state, which comprises the **occurrence of all** events;



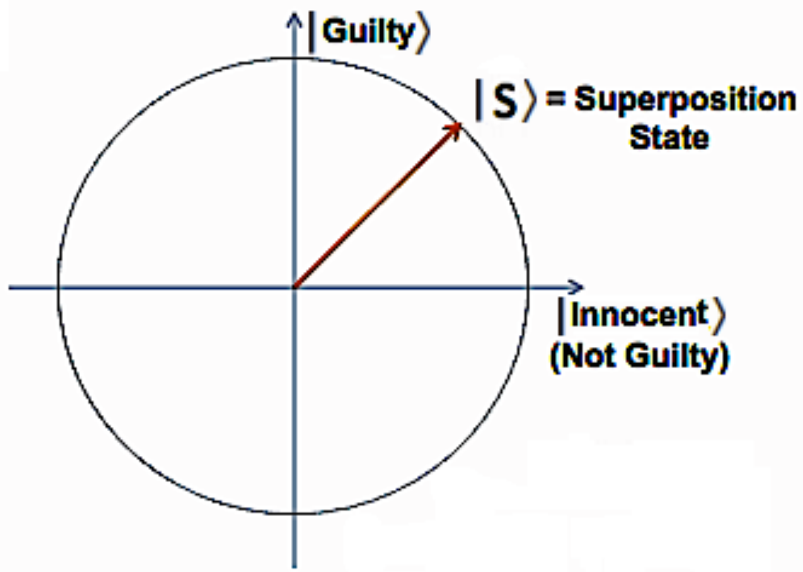
Classical vs Quantum: Events

The superposition state:

$$|S\rangle = \frac{e^{i\theta_{innocent}}}{\sqrt{2}}|Innocent\rangle + \frac{e^{i\theta_{guilty}}}{\sqrt{2}}|Guilty\rangle$$

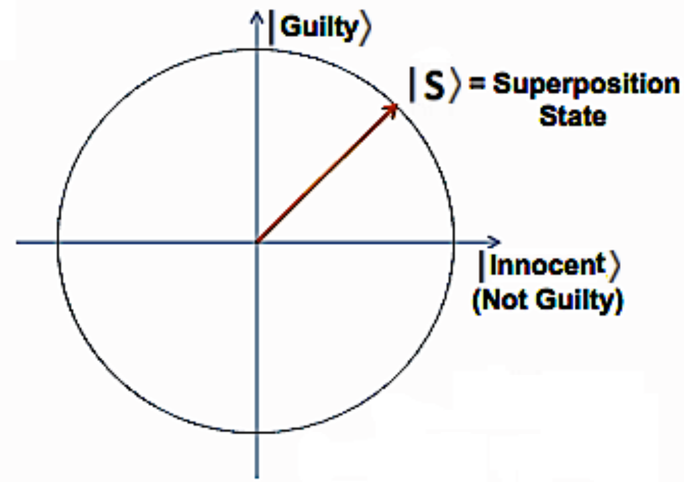
Quantum Normalization Axiom:

$$\left|\frac{e^{i\theta_{innocent}}}{\sqrt{2}}\right|^2 + \left|\frac{e^{i\theta_{guilty}}}{\sqrt{2}}\right|^2 = 1$$



Classical vs Quantum: Events

Quantum Normalization Axiom:



$$\left|\frac{e^{i\theta_{innocent}}}{\sqrt{2}}\right|^2 + \left|\frac{e^{i\theta_{guilty}}}{\sqrt{2}}\right|^2 = 1$$

$$= \left(\frac{e^{i\theta_{innocent}}}{\sqrt{2}}\right) \cdot \left(\frac{e^{i\theta_{innocent}}}{\sqrt{2}}\right)^* + \left(\frac{e^{i\theta_{guilty}}}{\sqrt{2}}\right) \cdot \left(\frac{e^{i\theta_{guilty}}}{\sqrt{2}}\right)^* =$$

$$= \left(\frac{e^{i\theta_{innocent} - e^{i\theta_{innocent}}}}{\sqrt{2}\sqrt{2}}\right) + \left(\frac{e^{i\theta_{guilty} - e^{i\theta_{guilty}}}}{\sqrt{2}\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

Classical vs Quantum: System State

CLASSICAL

- is a function that is responsible to assign a probability value to the outcome of an event.

- In our example, if nothing is told to the juror about the guiltiness or innocence of the defendant, then:

$$Pr(Guilty) = 0.5$$

Classical vs Quantum: System State

QUANTUM

- The system state is a unit-length N-dimensional vector, defined by a **superposition** state, that **maps events** into **probabilities;**
- The state is projected onto the subspaces corresponding to an event;
- The probability of the event corresponds to the squared length of this projection;

Classical vs Quantum: System State

QUANTUM

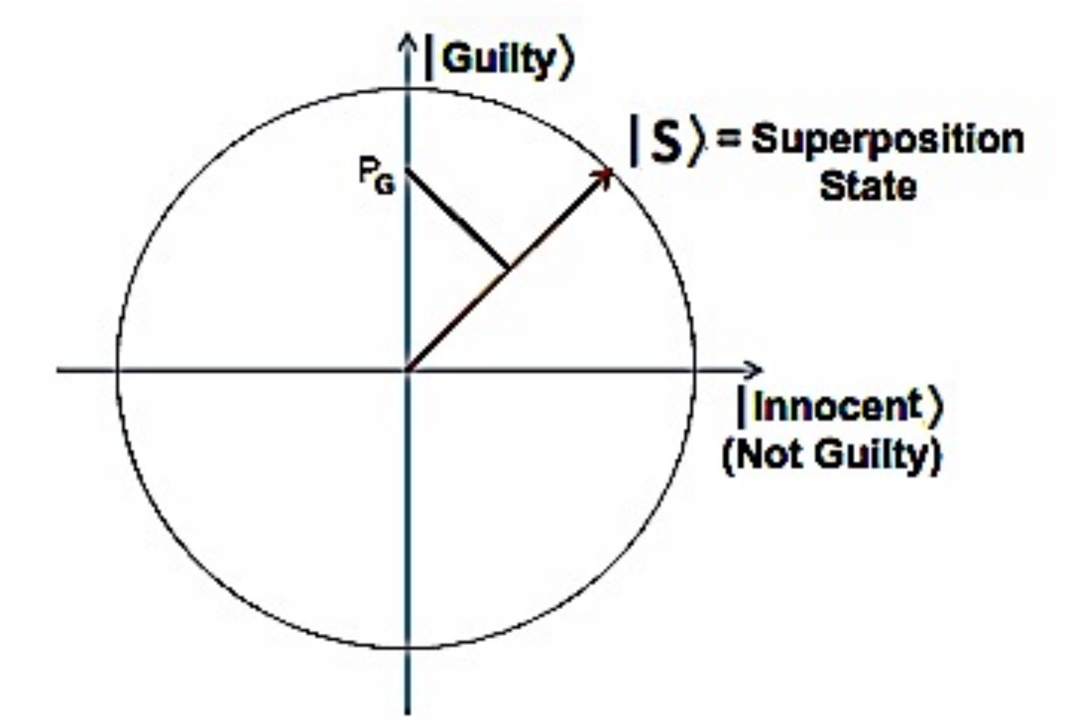
- The system state is a unit-length N-dimensional vector, defined by a **superposition** state, that **maps events** into **probabilities;**

$$|S\rangle = \frac{e^{i\theta_{innocent}}}{\sqrt{2}}|Innocent\rangle + \frac{e^{i\theta_{guilty}}}{\sqrt{2}}|Guilty\rangle$$

Classical vs Quantum: System State

QUANTUM

- The state is projected onto the subspaces corresponding to an event;



Classical vs Quantum: System State

QUANTUM

- The probability of the event corresponds to the squared length of this projection;

$$|S\rangle = \frac{e^{i\theta_{innocent}}}{\sqrt{2}}|Innocent\rangle + \frac{e^{i\theta_{guilty}}}{\sqrt{2}}|Guilty\rangle$$

$$Pr(|Guilty\rangle) = \left|\frac{e^{i\theta_{guilty}}}{\sqrt{2}}\right|^2 = \left(\frac{e^{i\theta_{guilty}}}{\sqrt{2}}\right) \cdot \left(\frac{e^{i\theta_{guilty}}}{\sqrt{2}}\right)^* = 0.5$$

Classical vs Quantum: State Revision

CLASSICAL

- An event is observed and we want to determine the probabilities after observing this fact

- Uses the conditional probability formula:

$$Pr(Innocent|Guilty) = \frac{Pr(Innocent \cap Guilty)}{Pr(Guilty)}$$

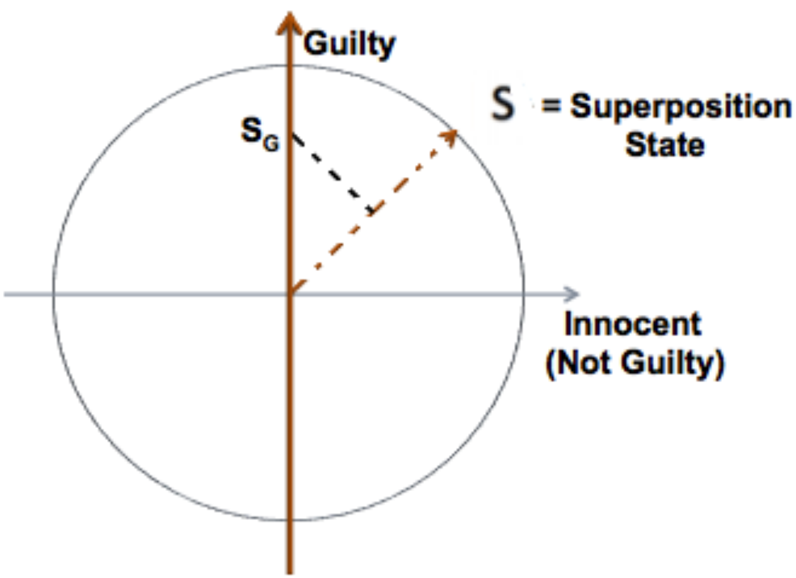
Classical vs Quantum: State Revision

QUANTUM

- Changes the original state vector by projecting the original state onto the subspace representing the observed event;

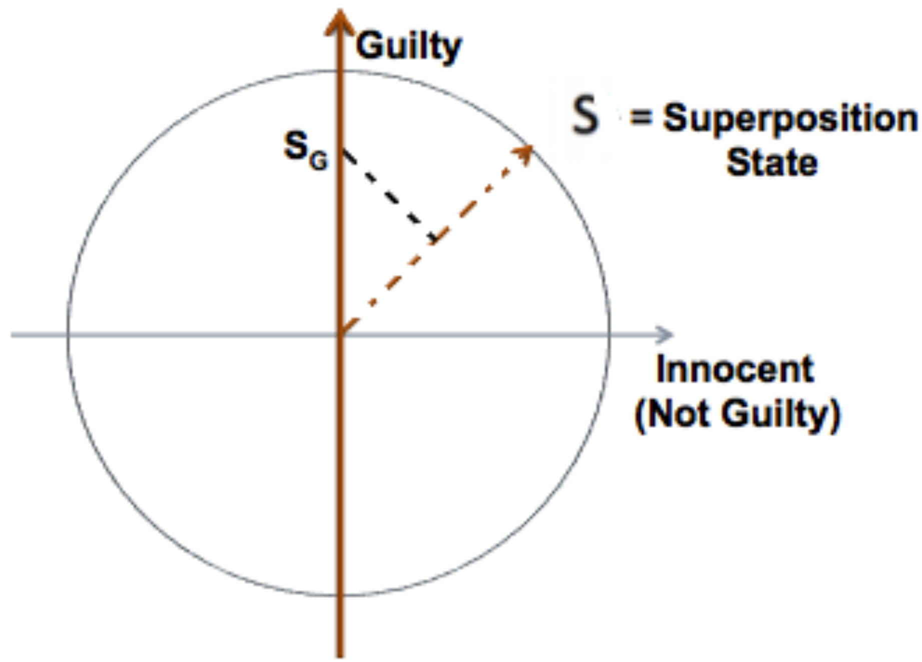
- The length of the projection is used as a normalization factor

$$|S_G\rangle = \frac{P_G|S\rangle}{||P_G|S\rangle||}$$



Classical vs Quantum: State Revision

QUANTUM STATE REVISION



$$|S_G\rangle = \frac{P_G|S\rangle}{||P_G|S\rangle||}$$

$$|S_G\rangle = \frac{(1/\sqrt{2})|Guilty\rangle}{\sqrt{(1/\sqrt{2})^2}}$$

$$|S_G\rangle = 1|Guilty\rangle + 0|Innocent\rangle$$

$$Pr(|Innocent\rangle) = 0^2 = 0$$

Classical Law of Total Probability

Suppose that events A_1, A_2, …, A_N form a set of mutually disjoint events, such that their union is all in the sample space for any other event B.

Then, the classical law of total probability can be formulated in the following way:

$$Pr(B) = \sum_{i=1}^{N} Pr(A_i) Pr(B|A_i) \qquad \text{where: } \sum_{i=1}^{N} A_i = 1$$

Quantum Law of Total Probability

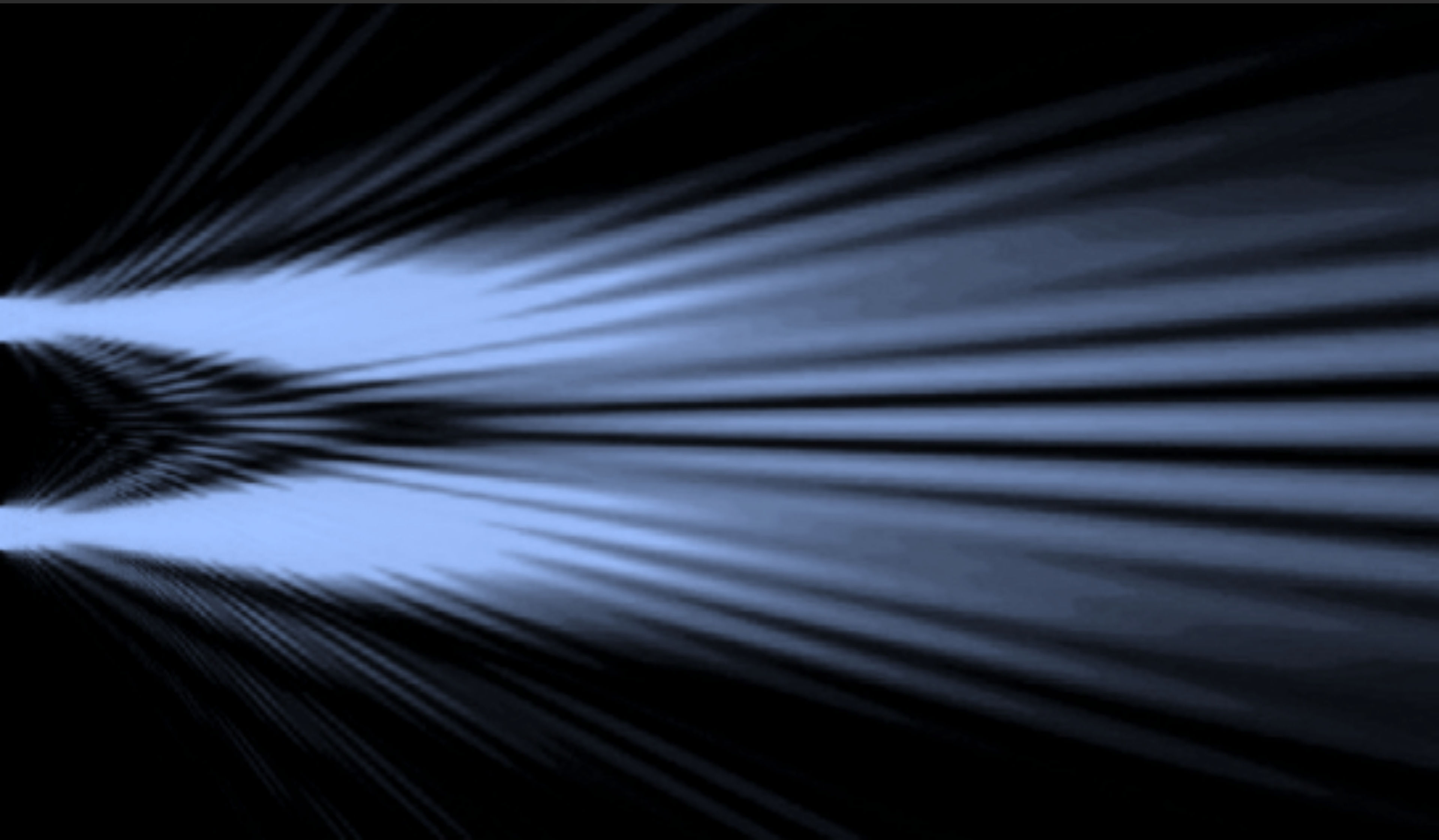
The quantum law of total probability can be derived by converting **classical probabilities** into **quantum amplitudes**!

BORN'S RULE: $Pr(A) = | \, e^{i\theta_A} \psi_A \, |^2$

The quantum law of total probability is given by:

$$Pr(B) = \left| \sum_{x=1}^{N} e^{i\theta_x} \psi_{A_x} \psi_{B|A_x} \right|^2 \qquad \sum_{x=1}^{N} \left| e^{i\theta_x} \psi_{A_x} \right|^2 = 1$$

Interference Effects



Interference Effects

Quantum law of total probability:

$$Pr(B) = \left| \sum_{x=1}^{N} e^{i\theta_x} \psi_{A_x} \psi_{B|A_x} \right|^2$$

Knowing that $\cos(\theta_1 - \theta_2) = \frac{e^{i\theta_1 - i\theta_2} + e^{i\theta_2 - i\theta_1}}{2}$, then…

$$Pr(B) = \sum_{i=1}^{n} \left| \psi_{A_i} \psi_{B|A_i} \right|^2$$ + *Interference*

Interference = $2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left| \psi_{A_i} \psi_{B|A_i} \right| \left| \psi_{A_j} \psi_{B|A_j} \right| \cos(\theta_i - \theta_j)$

Interference Effects

Quantum law of total probability:

$$Pr(B) = \left| \sum_{x=1}^{N} e^{i\theta_x} \psi_{A_x} \psi_{B|A_x} \right|^2$$

Knowing that $\frac{e^{i\theta_1 - i\theta_2} + e^{i\theta_2 - i\theta_1}}{2}$, then...

Classical Probability

$$Pr(B) = \sum_{i=1}^{n} \left| \psi_{A_i} \psi_{B|A_i} \right|^2$$

Quantum Interference

Interference = $2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left| \psi_{A_i} \psi_{B|A_i} \right| \left| \psi_{A_j} \psi_{B|A_j} \right| \cos(\theta_i - \theta_j)$

Interference Effects

The parameters generated in quantum interference effects grow at a very fast rate relatively to the number of unknown events.

The problem of automatically tune these parameters is still an open research question!

Num. Unobserved Nodes	Number of θ's	Num. Unobserved Nodes	Num. θ's
6	2016	11	2096128
7	8128	12	8386560
8	32640	13	33550336
9	130816	14	134209536
10	523776	15	536854528

Violations of Probability Theory

Violations of Probability Theory: Order of Effects

Q1. Do you generally think that **Passos Coelho** is honest and trustworthy? **(50%)**

Q2. How about **Ramalho Eanes**? **(60%)**

Q1. Do you generally think that **Ramalho Eanes** is honest and trustworthy? **(68%)**

Q2. How about **Passos Coelho?** **(57%)**

Violations of Probability Theory: Order of Effects

Q1. Do you generally think that **Passos Coelho** is honest and trustworthy? **(50%)**

18%

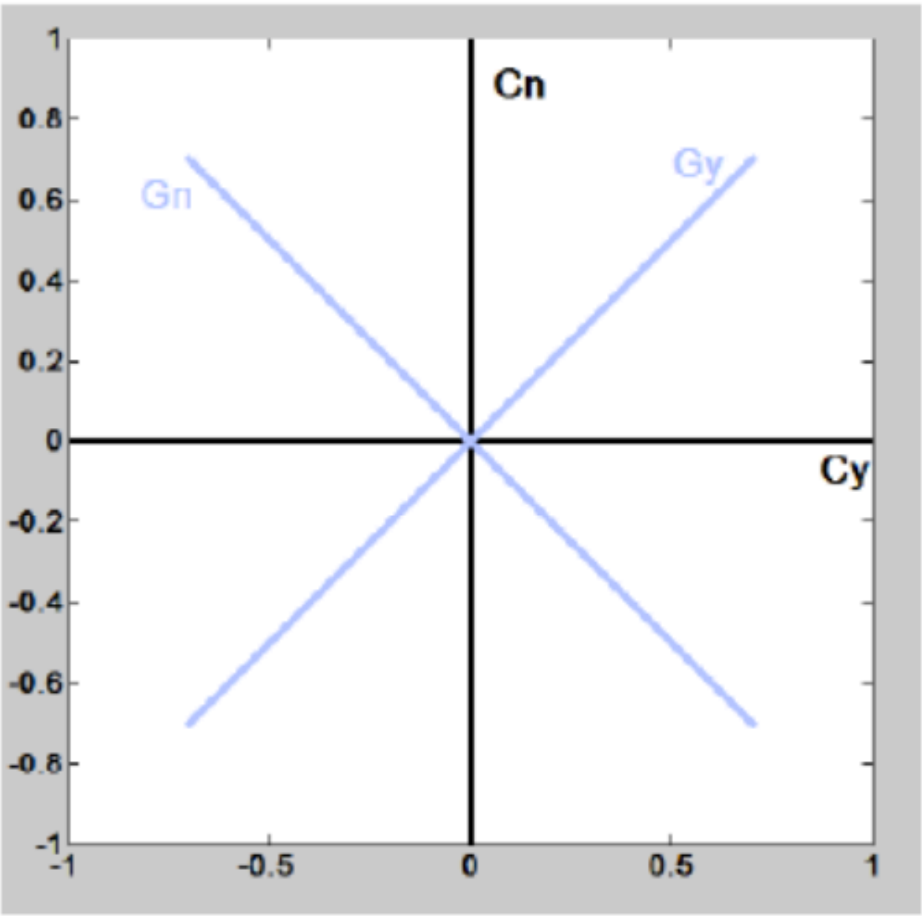
Q1. Do you generally think that **Ramalho Eanes** is honest and trustworthy? **(68%)**

Q2. How about **Ramalho Eanes?** **(60%)**

3 %

Q2. How about **Passos Coelho?** **(57%)**

Violations of Probability Theory: Order of Effects



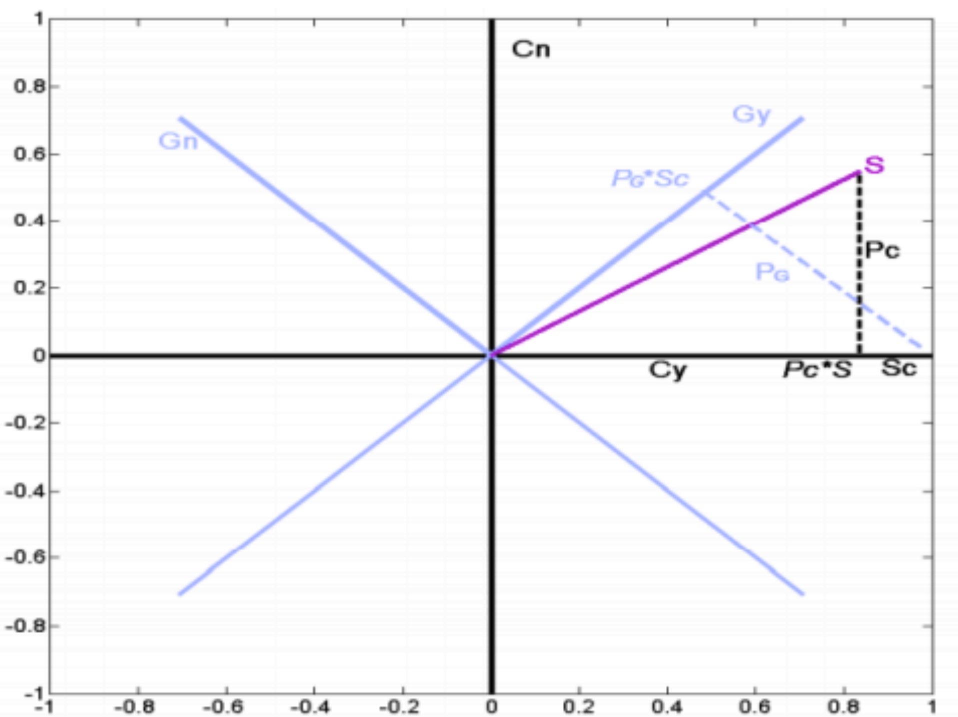
Cx Axis: Passos Coelho

Gx Axis: General Ramalho Eanes

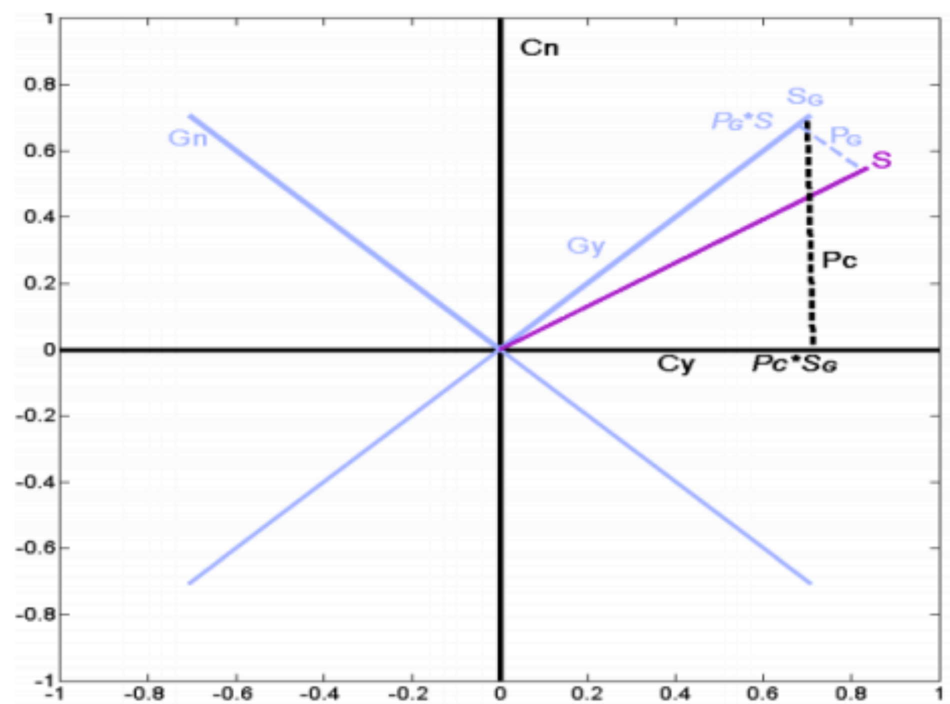
Violations of Probability Theory: Order of Effects

Using a quantum model, the probability of responses differ when asked first vs. when asked second.

Ramalho Eanes



Passos Coelho



Violations of Probability Theory: Order of Effects

Passos Coelho is a honest person:

$$|S\rangle = 0.8367|P\rangle + 0.5477\bar{P}\rangle$$

General Eanes is a honest person:

$$|S\rangle = 0.9789|G\rangle - 0.2043\bar{G}\rangle$$

Analysis of the first question – **Passos Coelho**

$$\Pr(Cy) = \|P_C|S\rangle\|^2 = |0.8367|^2 = 0.70$$
$$\Pr(Cn) = \|P_C|S\rangle\|^2 = |0.5477|^2 = 0.30$$

Violations of Probability Theory: Order of Effects

Passos Coelho is a honest person:

$$|S\rangle = 0.8367|P\rangle + 0.5477\bar{P}\rangle$$

General Eanes is a honest person:

$$|S\rangle = 0.9789|G\rangle - 0.2043\bar{G}\rangle$$

Analysis of the first question – **General Eanes**

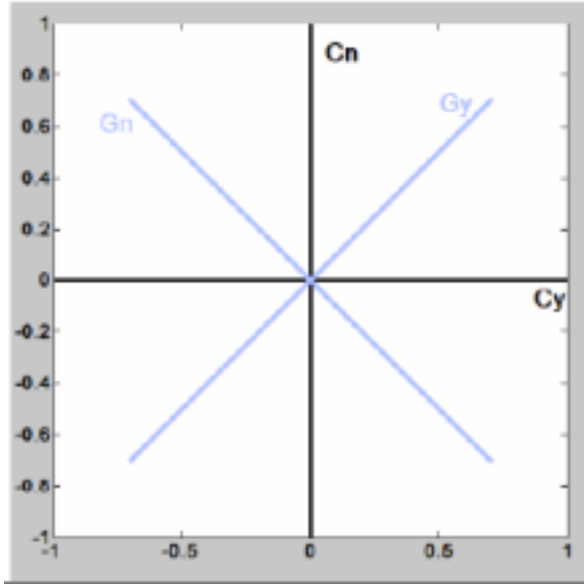
$$\Pr(Gy) = \|P_G|S\rangle\|^2 = |0.9789|^2 = 0.9582$$
$$\Pr(Gn) = \|P_G|S\rangle\|^2 = |-0.2043|^2 = 0.0417$$

Violations of Probability Theory: Order of Effects

Analysis of the **first question**:

$$Pr(Cy) = \|P_C|S\rangle\|^2 = |0.8367|^2 = 0.70$$
$$Pr(Cn) = \|P_C|S\rangle\|^2 = |0.5477|^2 = 0.30$$

$$Pr(Gy) = \|P_G|S\rangle\|^2 = |0.9789|^2 = 0.9582$$
$$Pr(Gn) = \|P_G|S\rangle\|^2 = |-0.2043|^2 = 0.0417$$



Analysis of the **second question**:

$$Pr(Cy) = (0.96).(0.50) + (0.04).(0.50) = 0.50$$

$$Pr(Gy) = (0.70).(0.50) + (0.30).(0.50) = 0.50$$

Violations of Probability Theory: Order of Effects

According to this simplified two-dimensional quantum model:

- **Large difference** between the agreement rates for two politicians in a **non-comparative context**: 70% for Passos Coelho and 96% for General Eanes

- There is **no difference** in the **comparative context**: 50% for both

Violations of Probability Theory: Order of Effects

Violations of Probability Theory: Order of Effects

315 active doctors were asked to estimate the **probability** that a specific patient had a **urinary tract infection** (UTI) given the patient's **history** and **physical examination along with laboratory data**

Violations of Probability Theory: Order of Effects

The physicians were divided into two groups:

- One receiving the **history** and **physical examination information first** (H&P-first)

- The other receiving the **laboratory data first** (H&P-last).

Violations of Probability Theory: Order of Effects

Results:

	H & P First	H & P Last
Prior Probability	$Pr(UTI) = 0.6740$	$Pr(UTI) = 0.6780$
First Set of Evidences	$Pr(UTI \mid H\&P) = 0.778$	$Pr(UTI \mid Lab) = 0.4400$
Final Set of Evidences	$Pr(UTI \mid H\&P, Lab) = 0.5090$	$Pr(UTI \mid Lab, H\&P) = 0.5910$

Classical probability fails to explain this, because:

$$p(H \mid A \cap B) = p(H \mid A) \cdot \frac{p(B \mid H \cap A)}{p(B \mid A)} = p(H \mid B) \cdot \frac{p(A \mid H \cap B)}{p(A \mid B)} = p(H \mid B \cap A)$$

Violations of Probability Theory: Order of Effects

In Trueblood & Busemeyer (2011) the authors proposed a quantum model to simulate the previous results.

- They project the initial superposition state into the subspace representing the observed event

- then they compute the squared modulus of this projection to extract the probabilities

Violations of Probability Theory: The Sure Thing Principle

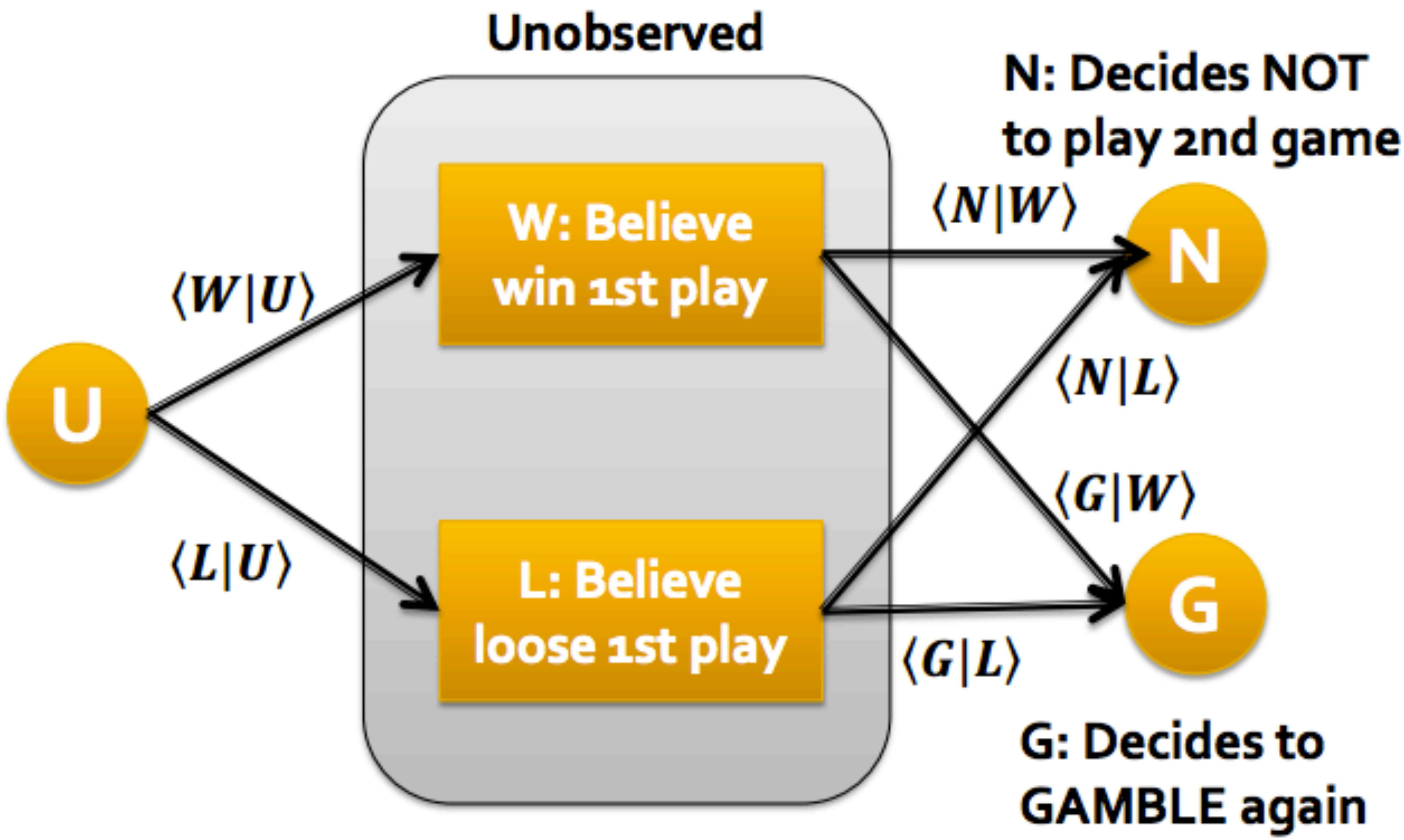
Violations of Probability Theory: The Two Stage Gambling Game

Participants were asked to **play a gambling game** that has an equal chance of **winning** $200 or **loosing** $100. **Three** conditions were verified:

- **Informed** that they **won** the 1^{st} gamble;
- **Informed** that they **lost** the 1^{st} gamble;
- **Did not know** if they won or lost the 1^{st} gamble;

A Tversky and E Shafir, E. (1992), 'The disjunction effect in choice under uncertainty', Journal of Psychological Science 3, 305–309

Violations of Probability Theory: The Two Stage Gambling Game



Violations of Probability Theory: The Two Stage Gambling Game

Results

Results from the two-stage gambling task

Study	Known win	Known loss	Unknown	N^a
Tversky & Shafir	0.69	0.58	0.37	169
Tversky & Shafir[b]	0.73	0.63	0.79	144
Kühberger *et al.*	0.72	0.47	0.48	188
Lambdin & Burdsal	0.63	0.45	0.41	165
Average	0.68	0.50	0.42	
Quantum model	0.72	0.52	0.38	

$^a N$ refers to the number of choices included in each proportion.

Violations of Probability Theory: The Two Stage Gambling Game

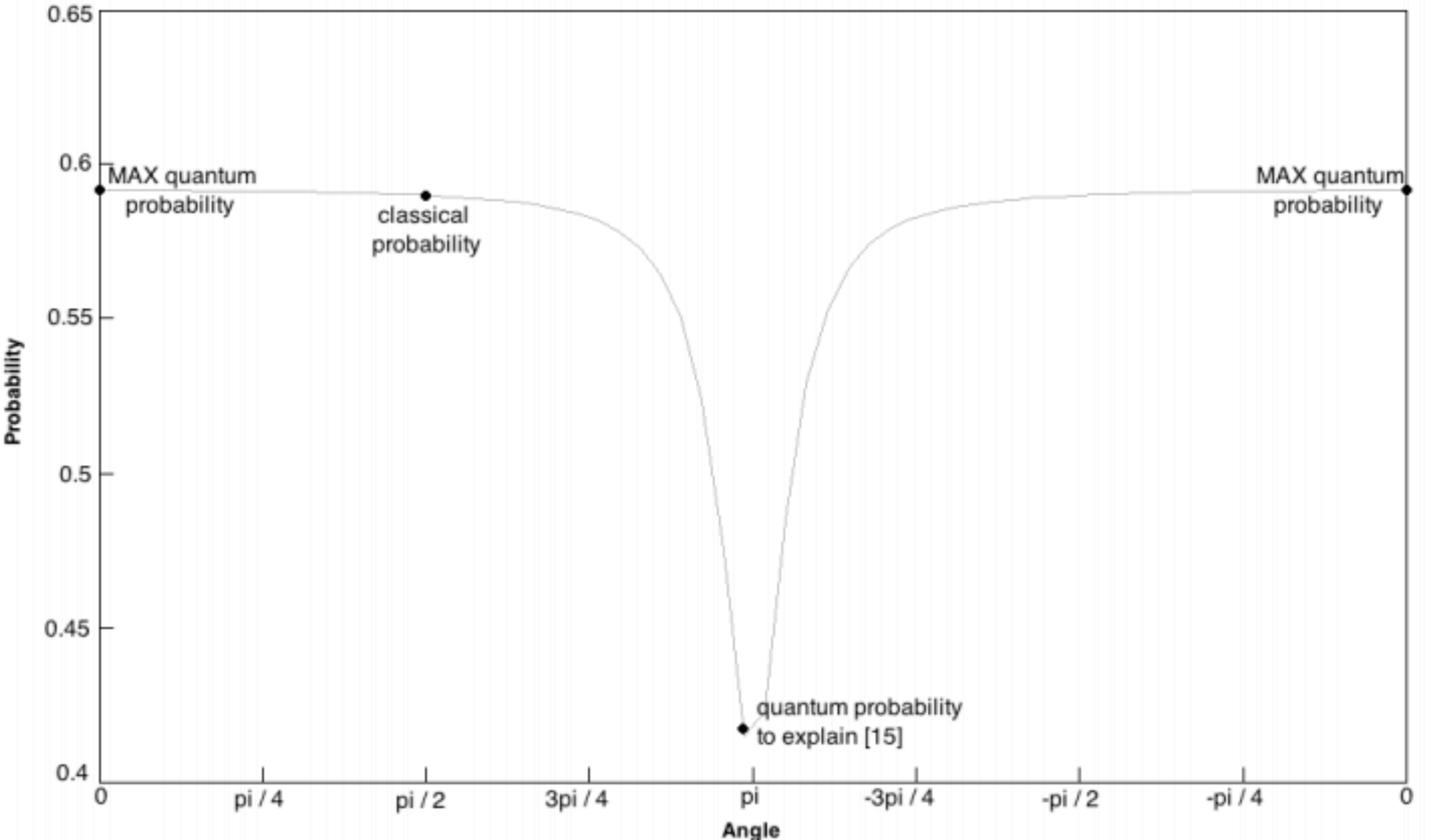
Quantum model:

Law of total amplitude:

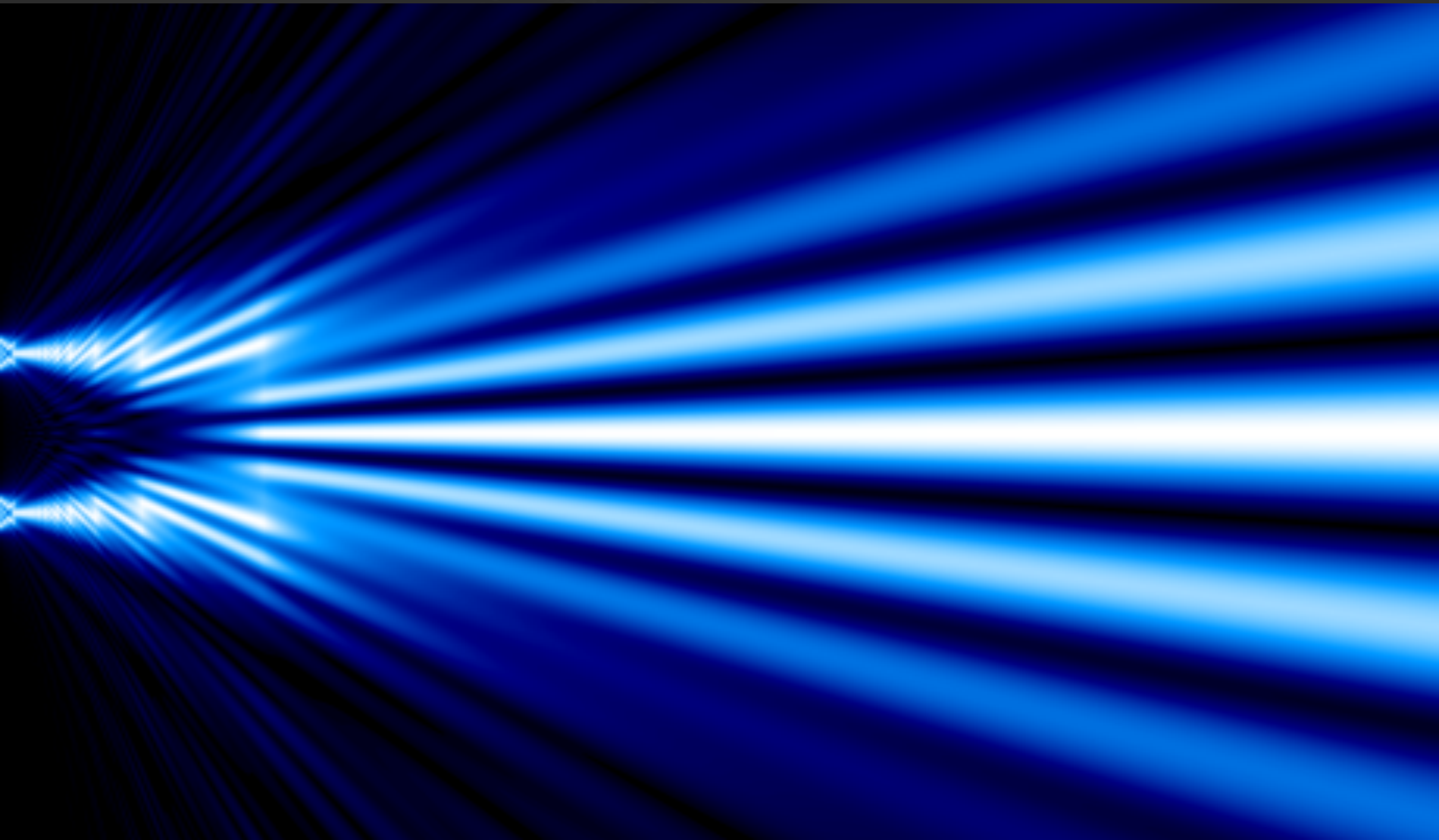
$$\Pr(\langle G|U\rangle) = |\langle W|U\rangle\langle G|W\rangle + \langle L|U\rangle\langle G|L\rangle|^2$$

$$= |\langle W|U\rangle\langle G|W\rangle|^2 + |\langle L|U\rangle\langle G|L\rangle|^2 +$$
$$+2. Re[\langle W|U\rangle\langle G|W\rangle\langle L|U\rangle\langle G|L\rangle. Cos\ \theta]$$

Violations of Probability Theory: The Two Stage Gambling Game



Violations of Probability Theory: The Double Slit Experiment



The Double Slit Experiment

- A single electron is dispersed from a light source

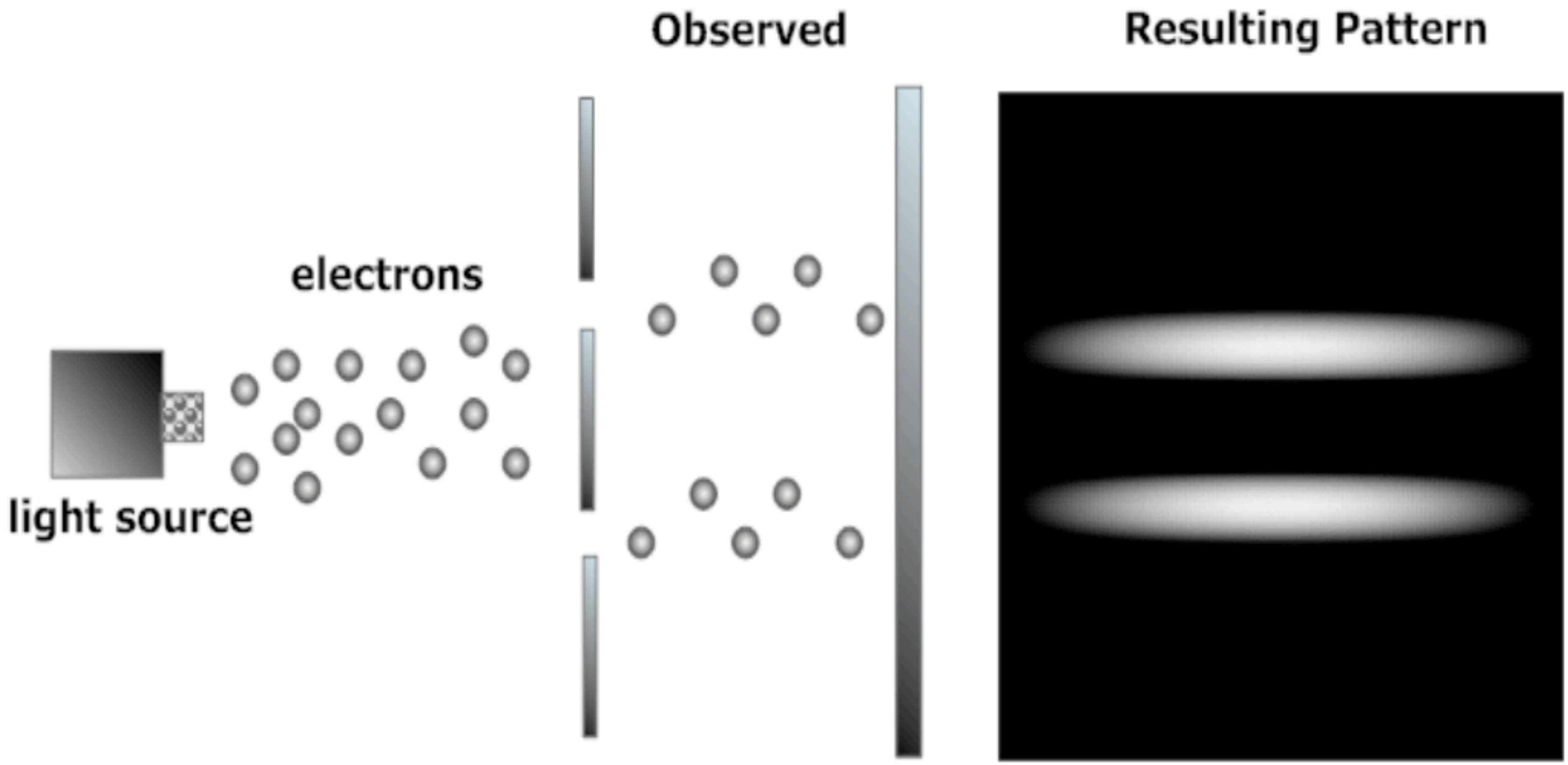
- The electron must pass through one of two channels (C1 or C2) from which it can reach one of the two detectors (D1 or D2).

The Double Slit Experiment

- Two conditions are examined:
  - The channel through which the electron passes in is **observed**.
  - The channel through which the electron passes in is **not observed.**

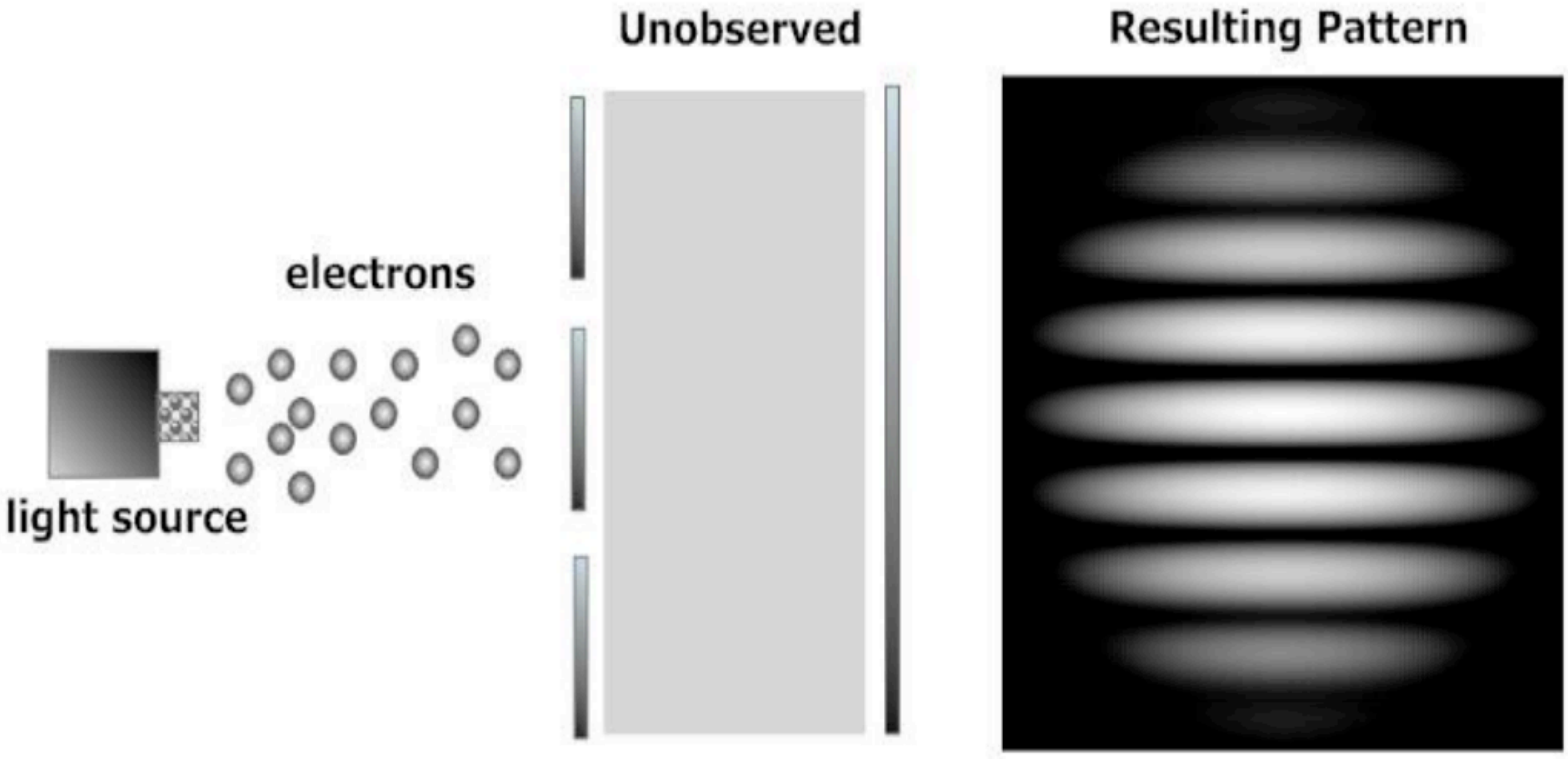
The Double Slit Experiment

The results (when **observing**):

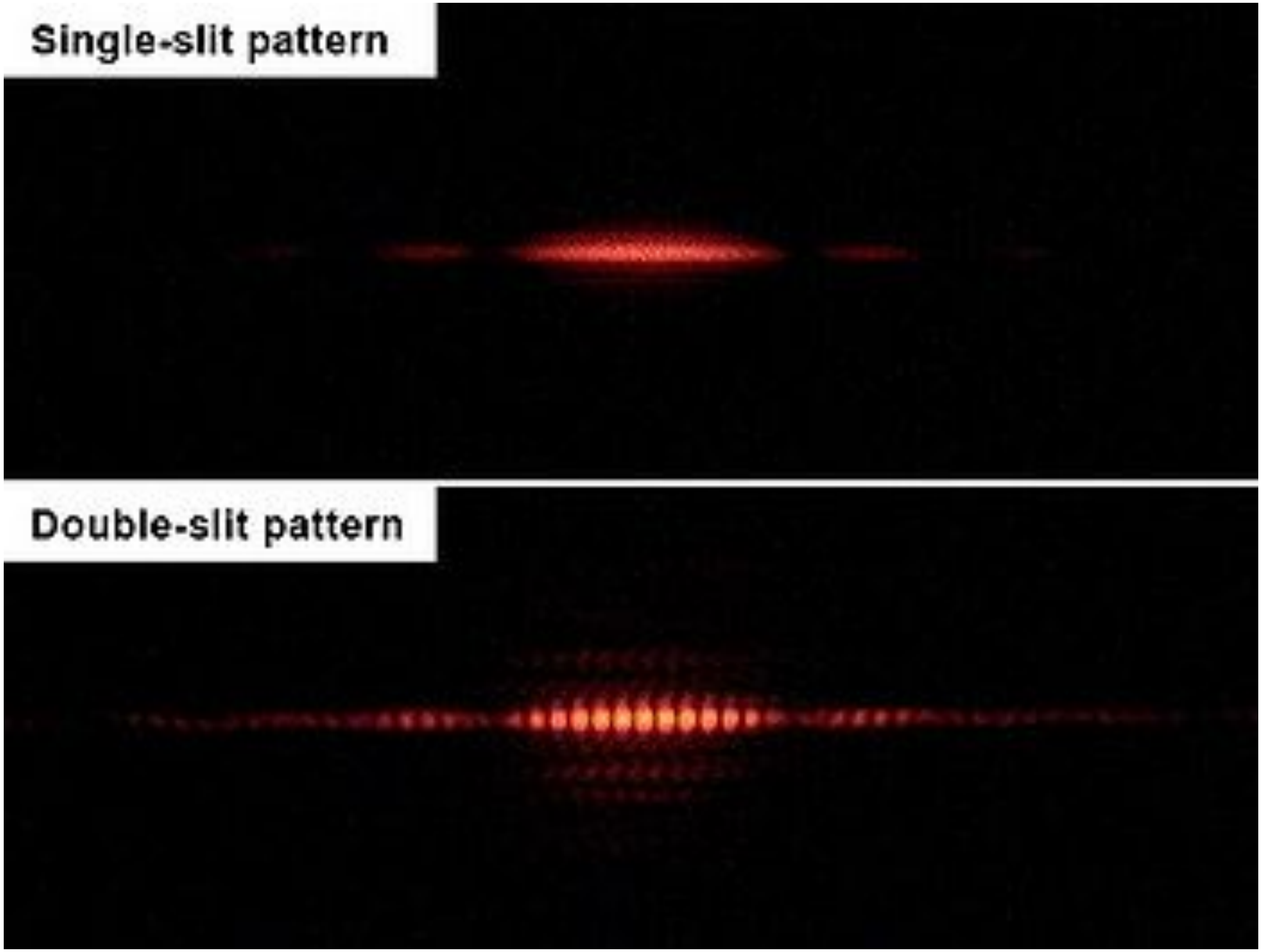


The Double Slit Experiment

The results (when **not observing**):

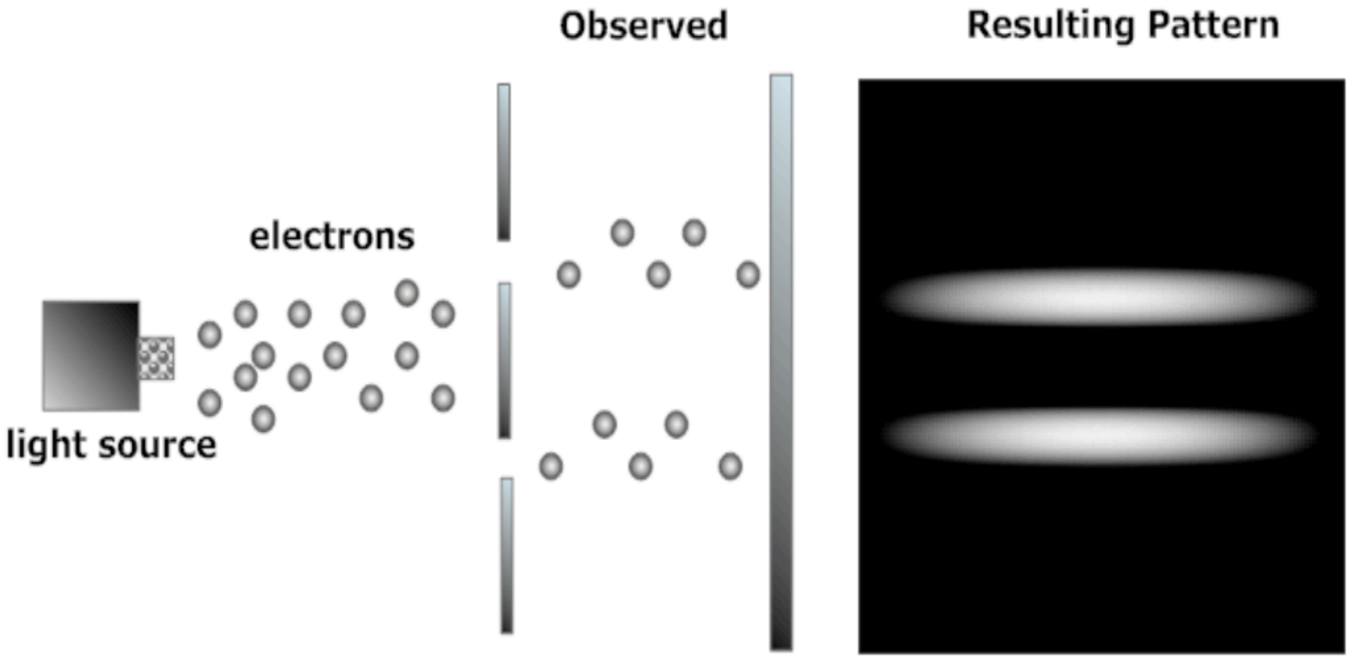


The Double Slit Experiment

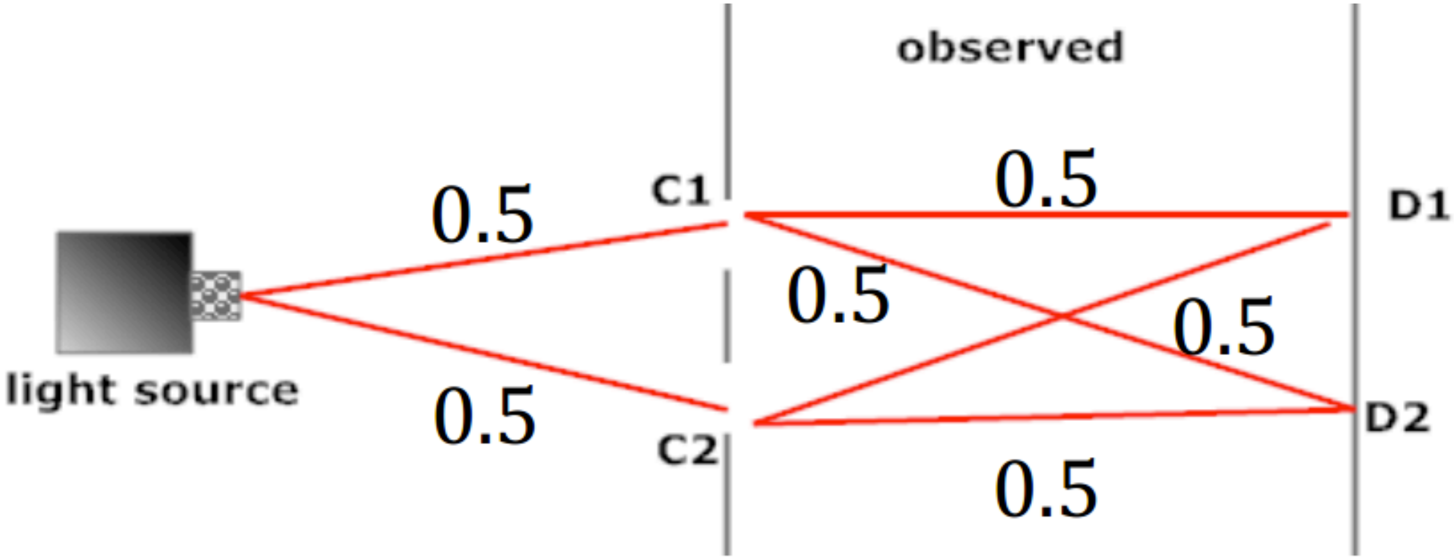


The Double Slit Experiment

- Let's analyze the first condition (when the path of the electron was observed)!



The Double Slit Experiment (Classical Probability)

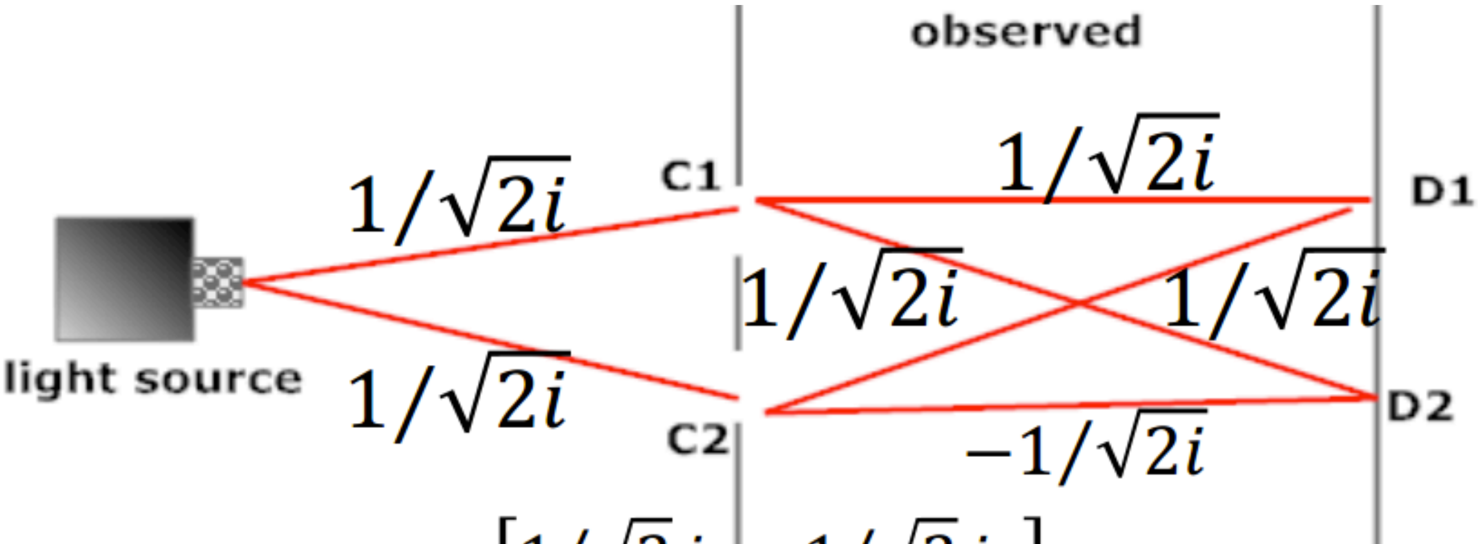


$$Pr(c1) = [0.5 \quad 0]\begin{bmatrix}0.5 & 0.5\\0.5 & 0.5\end{bmatrix} = [0.25 \quad 0.25]$$

$$\Pr(c2) = [0 \quad 0.5]\begin{bmatrix}0.5 & 0.5\\0.5 & 0.5\end{bmatrix} = [0.25 \quad 0.25]$$

$$\Pr(c1 \ or \ c2) = \Pr(c1) + \Pr(c2) = [0.5 \quad 0.5]$$

The Double Slit Experiment (Quantum Probability)



$$c1 = \begin{bmatrix}1/\sqrt{2}\,i & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\,i & 1/\sqrt{2}\,i \\ 1/\sqrt{2}\,i & -1/\sqrt{2}\,i\end{bmatrix} = \begin{bmatrix}-0.5 & -0.5\end{bmatrix}$$

$$c2 = \begin{bmatrix}0 & 1/\sqrt{2}\,i\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\,i & 1/\sqrt{2}\,i \\ 1/\sqrt{2}\,i & -1/\sqrt{2}\,i\end{bmatrix} = \begin{bmatrix}-0.5 & -0.5\end{bmatrix}$$

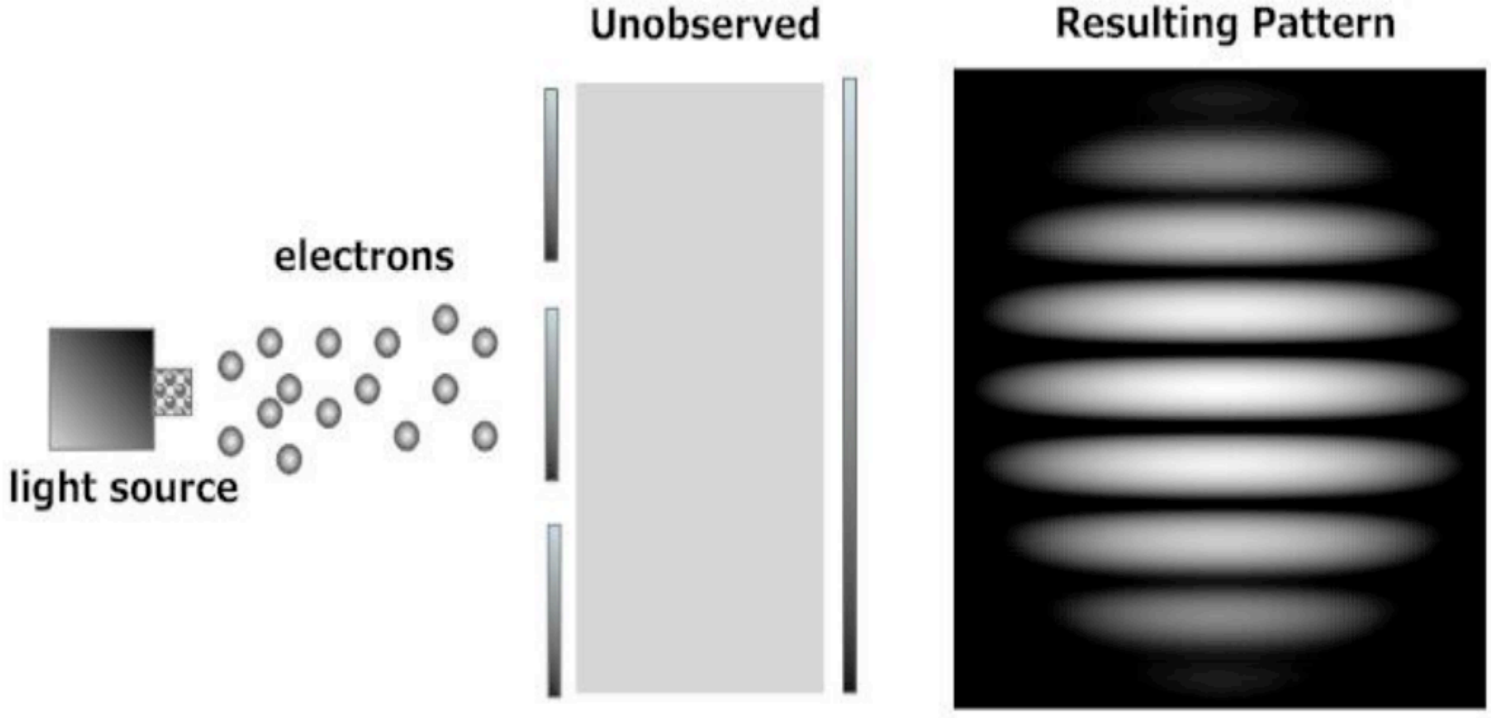
$$\Pr(c1\ or\ c2) = c1^2 + c2^2 = \begin{bmatrix}0.5 & 0.5\end{bmatrix}$$

The Double Slit Experiment

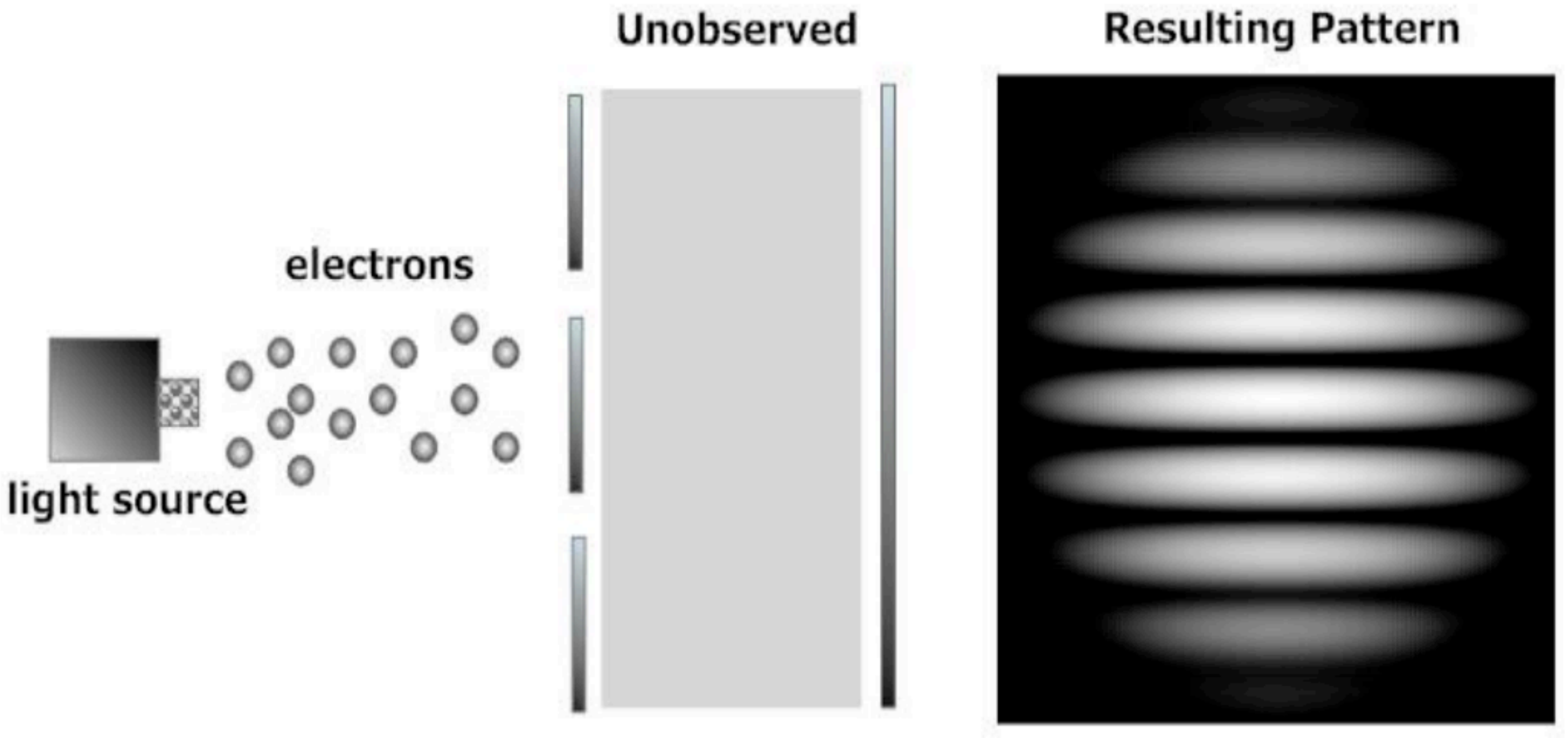
When the paths **were observed**, both classical and quantum probabilities obtained the **same** results!

The Double Slit Experiment

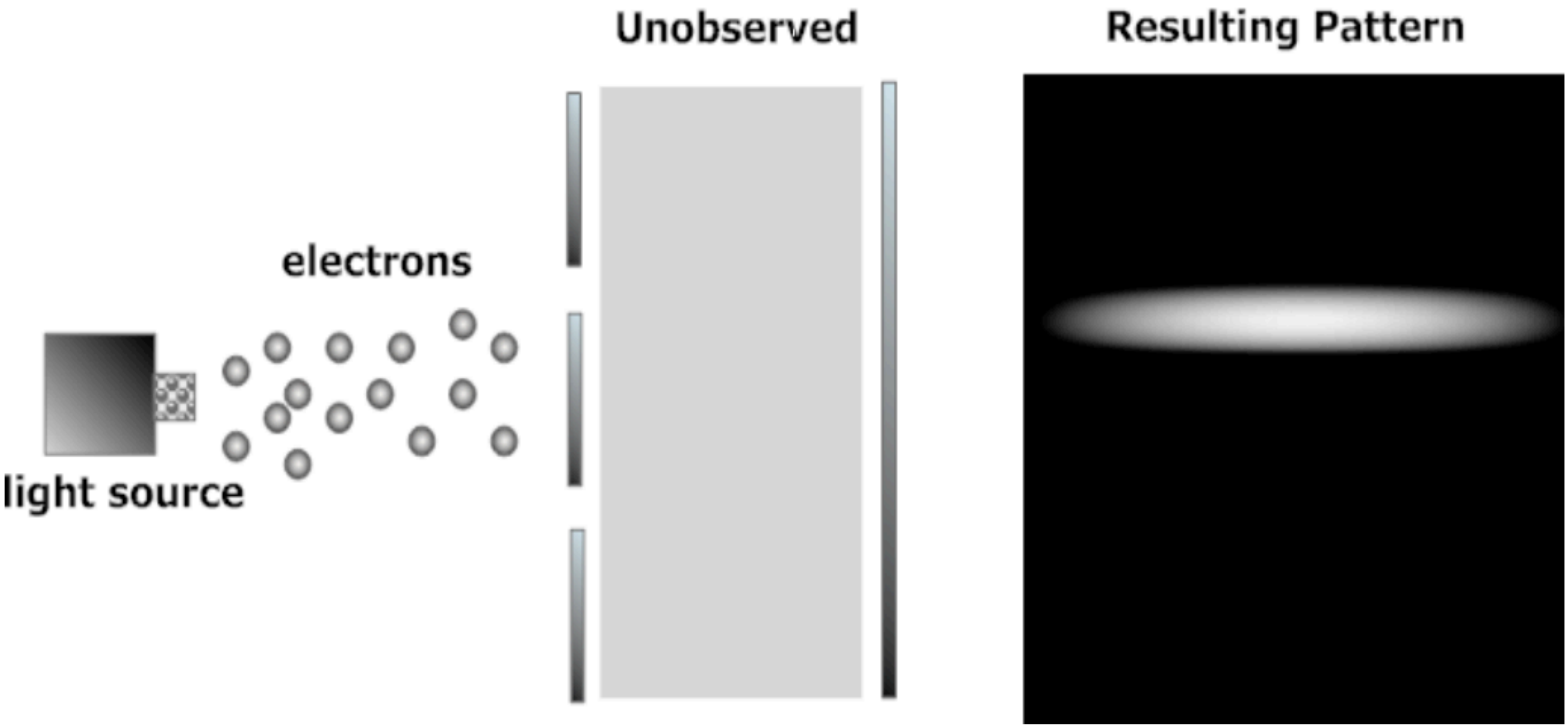
- Let's analyze the second condition (when the path of the electron was **not observed**)!



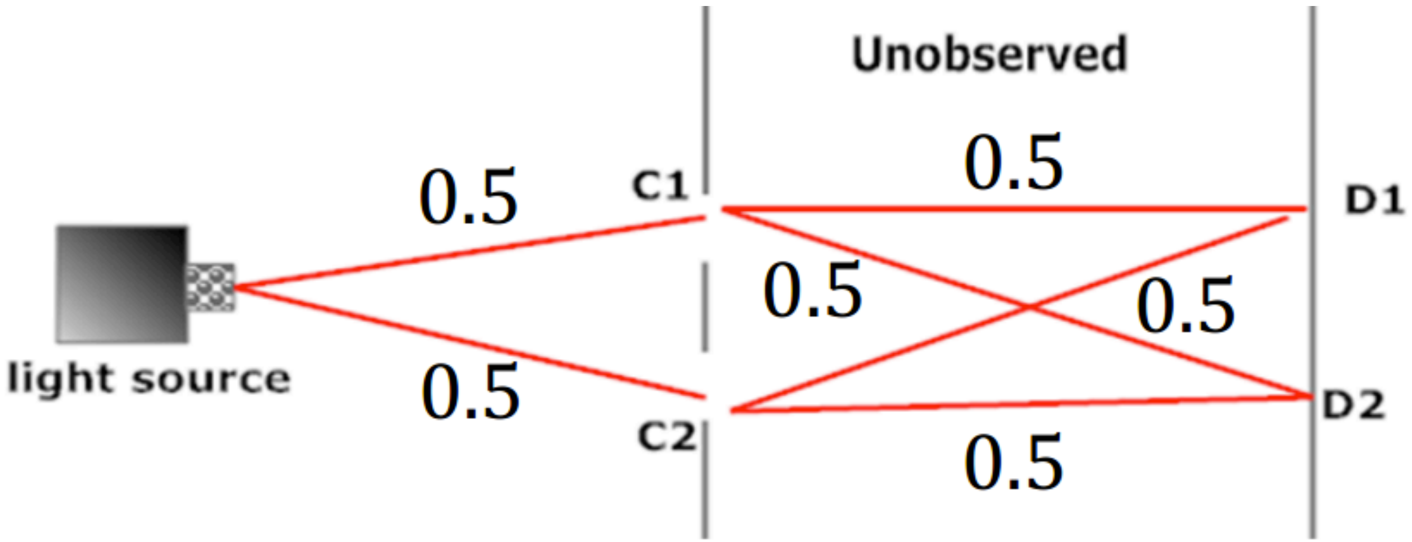
The Double Slit Experiment



The Double Slit Experiment



The Double Slit Experiment (Classical Probability)

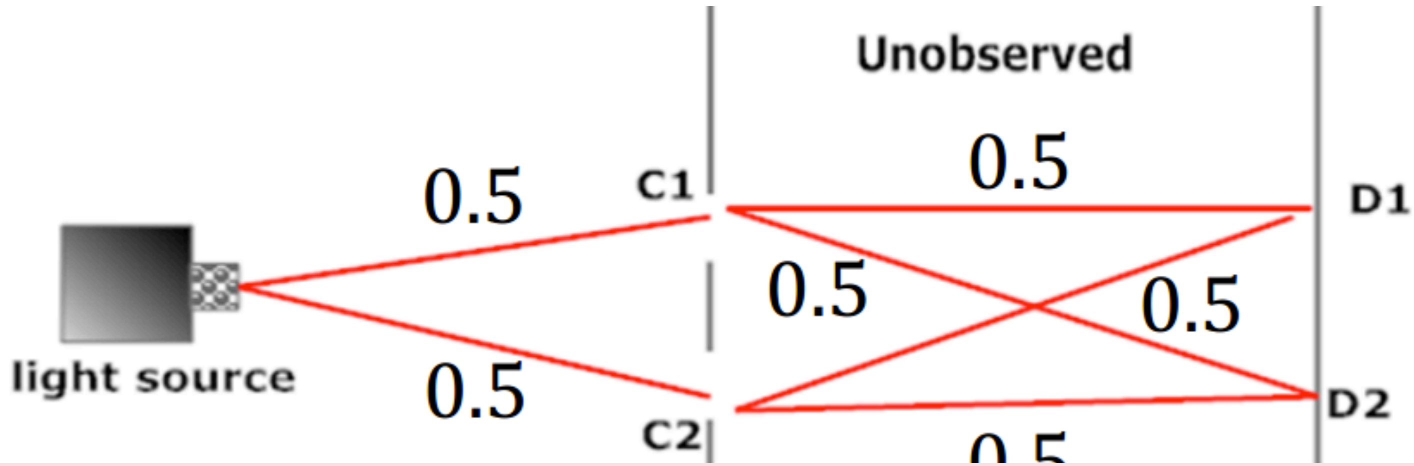


$$Pr(c1) = \begin{bmatrix}0.5 & 0\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0.5 & 0.5\end{bmatrix} = \begin{bmatrix}0.25 & 0.25\end{bmatrix}$$

$$\Pr(c2) = \begin{bmatrix}0 & 0.5\end{bmatrix}\begin{bmatrix}0.5 & 0.5\\0.5 & 0.5\end{bmatrix} = \begin{bmatrix}0.25 & 0.25\end{bmatrix}$$

$$\Pr(c1\ or\ c2) = \Pr(c1) + \Pr(c2) = \begin{bmatrix}0.5 & 0.5\end{bmatrix}$$

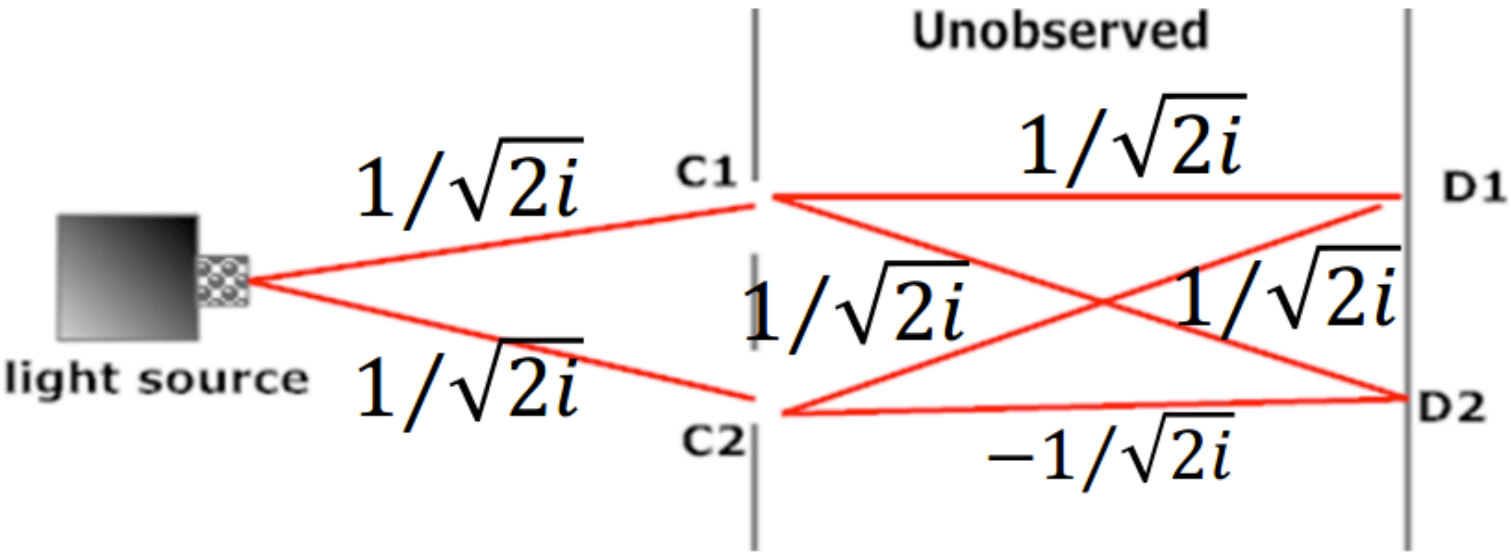
The Double Slit Experiment (Classical Probability)



Cannot Explain the Interference Pattern!

$$\Pr(c1\ or\ c2) = \Pr(c1) + \Pr(c2) = [0.5 \quad 0.5]$$

The Double Slit Experiment (Quantum Probability)



$$c12 = \begin{bmatrix} 1/\sqrt{2}\,i & 1/\sqrt{2}\,i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2}\,i & 1/\sqrt{2}\,i \\ 1/\sqrt{2}\,i & -1/\sqrt{2}\,i \end{bmatrix} = \begin{bmatrix} -1 & 0 \end{bmatrix}$$

$$\Pr(c12) = c12^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

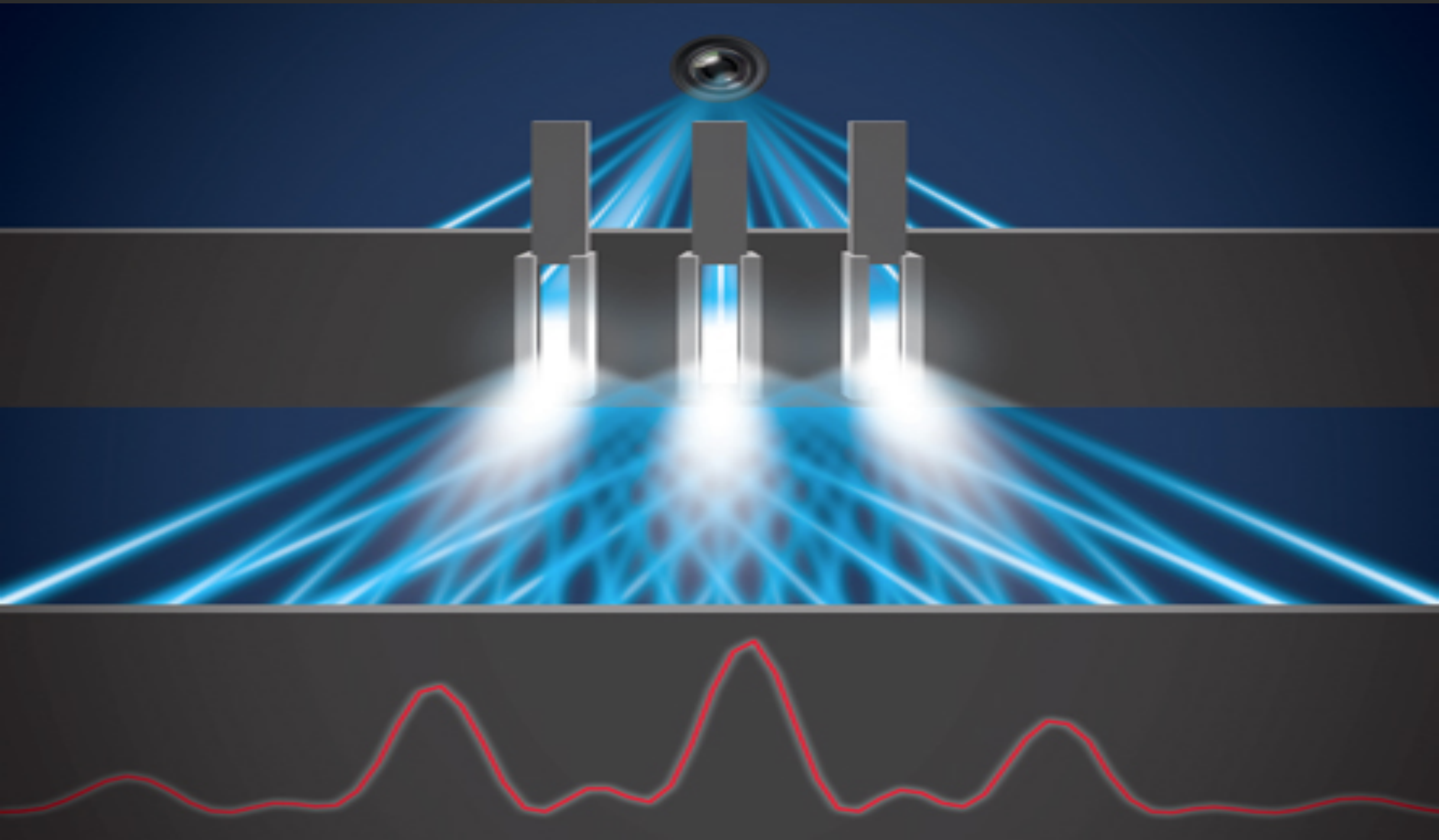
The Double Slit Experiment

- If we **do not observe** the system
- Then, we cannot assume that one of only two possible paths are taken

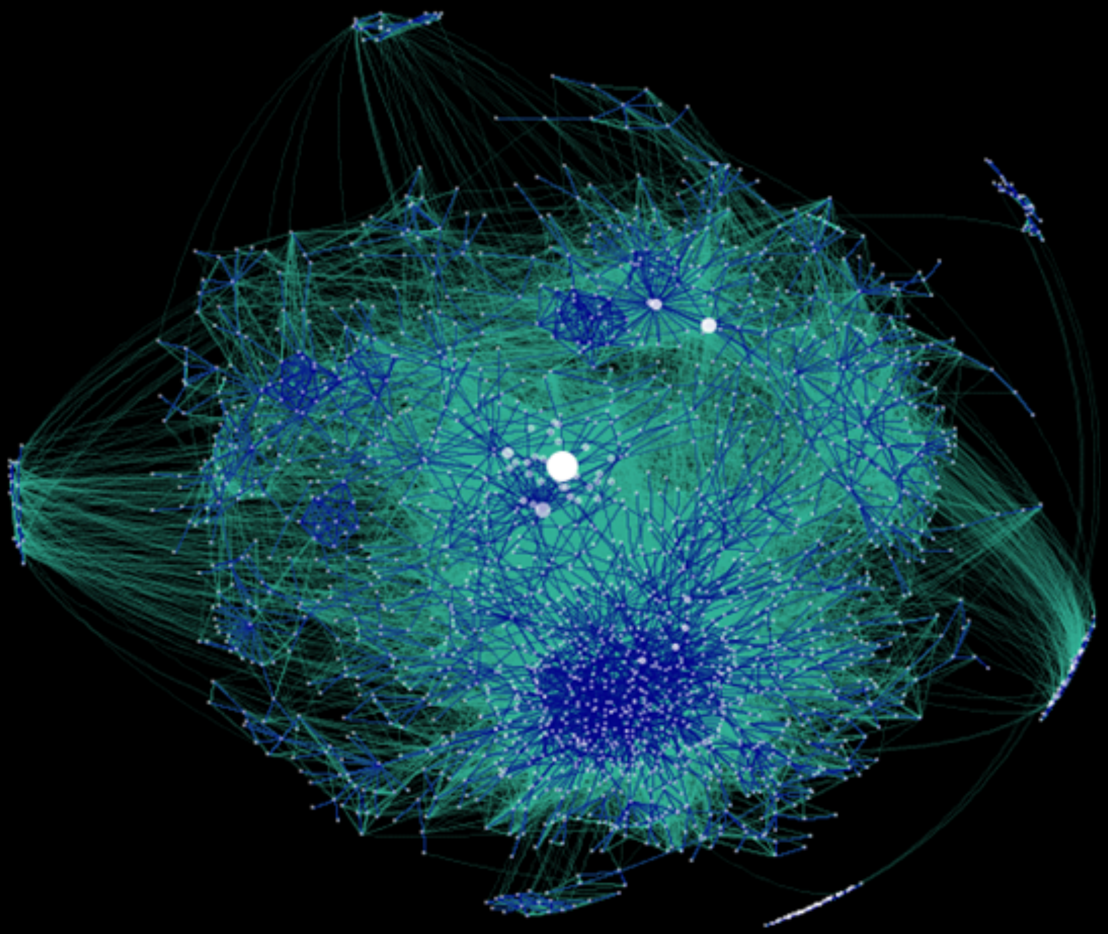
- In Quantum theory, the state is superposed between the two possible paths!

- **Quantum Theory REJECTS the single path trajectory principle**

Time to See the Double Slit Effect!!!



Bayesian Networks



Bayesian Networks

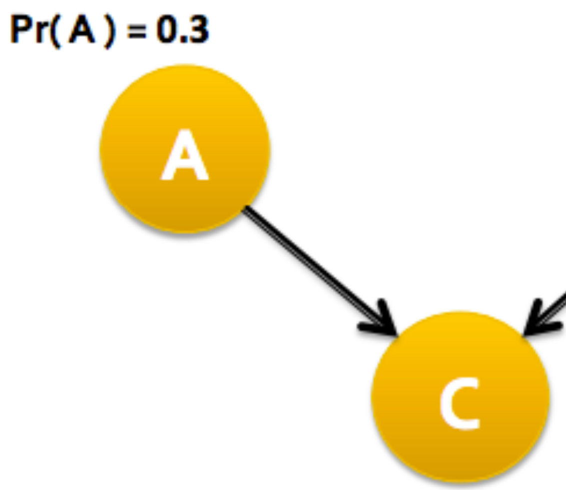
*Directed acyclic graph structure in which each **node** represents a different **random variable** and each **edge** represents **a direct causal influence** from source node to the target node.*

Bayesian Networks

*The graph represents **independence relationships** between variables and each node is associated with a **conditional probability table***

Bayesian Networks – Classical Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?



A	B	P(C | A B)
T	T	0.8
T	F	0.7
F	T	0.4
F	F	0.02

Bayesian Networks – Classical Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?

Exact inference in classical Bayesian Networks:

$$Pr_c(X|e) = \alpha Pr_c(X,e) = \alpha \left[\sum_{y \in Y} Pr_c(X,e,y) \right]$$

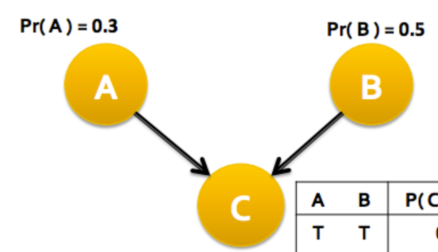
$$\text{Where} \quad \alpha = \frac{1}{\sum_{x \in X} Pr_c(X = x, e)}$$

Bayesian Networks – Classical Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?

$$Pr(C = t|A = t, B) =$$

$$Pr(A = t) \sum_{b \in B} Pr(B = b) Pr(C = t|A = t, B = b)$$



A	B	P(C | A B)
T	T	0.8
T	F	0.7
F	T	0.4
F	F	0.02

Bayesian Networks – Classical Inference

Full Joint probability distribution:

A	B	C	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12
T	T	F	0.3 X 0.5 X 0.2 = 0.03
T	F	T	0.3 X 0.5 X 0.7 = 0.105
T	F	F	0.3 X 0.5 x 0.3 = 0.045
F	T	T	
F	T	F	
F	F	T	
F	F	F	

We don't need to compute the entries where A is False!

Bayesian Networks – Classical Inference

Full Joint probability distribution and **normalize**:

A	B	C	Pr(A, B, C)	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12	0.4
T	T	F	0.3 X 0.5 X 0.2 = 0.03	0.1
T	F	T	0.3 X 0.5 X 0.7 = 0.105	0.35
T	F	F	0.3 X 0.5 x 0.3 = 0.045	0.15
	Sum		0.3	1

Bayesian Networks – Classical Inference

Full Joint probability distribution and **normalize**:

Just sum the entries where **C = T**

A	B	C	Pr(A, B, C)	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12	0.4
T	T	F	0.3 X 0.5 X 0.2 = 0.03	0.1
T	F	T	0.3 X 0.5 X 0.7 = 0.105	0.35
T	F	F	0.3 X 0.5 x 0.3 = 0.045	0.15

Bayesian Networks – Classical Inference

Full Joint probability distribution and **normalize**:

Just sum the entries where C = T

Pr(C = t | A = t, B) = 0.75

A	B	C	Pr(A, B, C)	Pr(A, B, C)
T	T	T	0.3 X 0.5 X 0.8 = 0.12	0.4
T	T	F	0.3 X 0.5 X 0.2 = 0.03	0.1
T	F	T	0.3 X 0.5 X 0.7 = 0.105	0.35
T	F	F	0.3 X 0.5 x 0.3 = 0.045	0.15

A Model for Quantum Bayesian Networks with Interference Effects

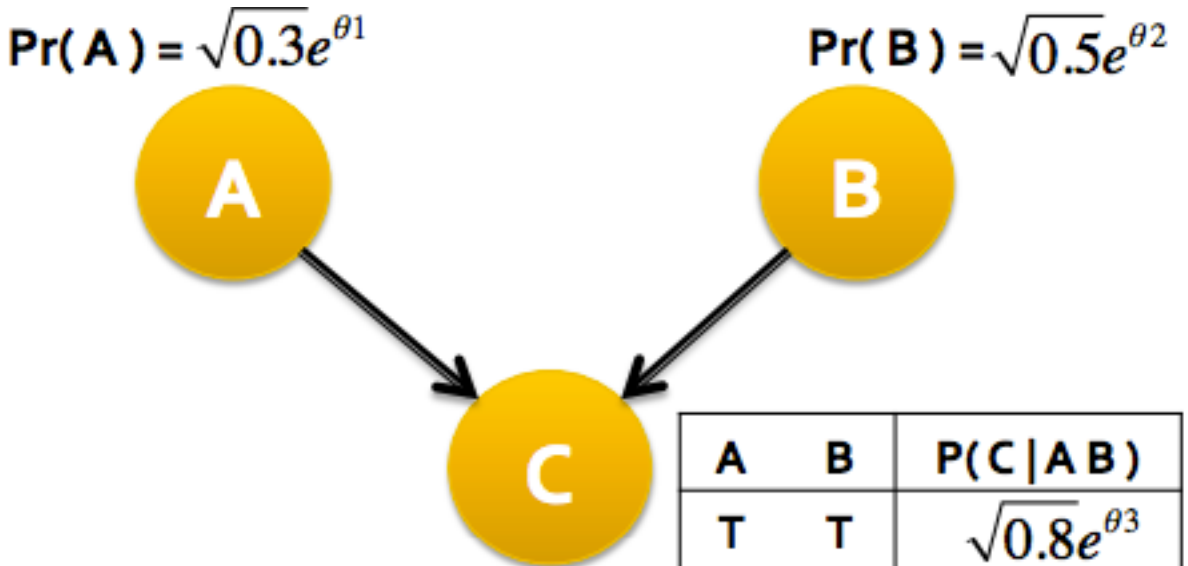
Bayesian Networks – Quantum Inference

Given a normal Bayesian Network, the first step of the model is to compute complex amplitudes out of real values using **Born's rule**!

$$Pr(A) = | \, e^{i\theta_A} \psi_A \, |^2$$

Bayesian Networks – Quantum Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?



A	B	P(C|AB)
T	T	$\sqrt{0.8}e^{\theta 3}$
T	F	$\sqrt{0.7}e^{\theta 4}$
F	T	$\sqrt{0.4}e^{\theta 5}$
F	F	$\sqrt{0.02}e^{\theta 6}$

Bayesian Networks – Quantum Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?

Exact inference in quantum Bayesian Networks:

$$Pr_q(X|e) = \alpha \sum_{i=1}^{|Y|} \left| \prod_{x}^{N} QPr(X_x|Parents(X_x), e, y=i) \right|^2 +$$

$$+2 \sum_{i=1}^{|Y|-1} \sum_{j=i+1}^{|Y|} \left| \prod_{x}^{N} QPr(X_x|Parents(X_x), e, y=i) \right| \left| \prod_{x}^{N} QPr(X_x|Parents(X_x), e, y=j) \right| \cos(\theta_i - \theta_j)$$

Bayesian Networks – Quantum Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?

The full joint probability distribution corresponds to the superposition state:

$$|S_A\rangle = \sqrt{0.3464}e^{\theta_1}|ABC\rangle + \sqrt{0.1732}e^{\theta_2}|AB\bar{C}\rangle +$$

$$+\sqrt{0.3240}e^{\theta_3}|A\bar{B}C\rangle + \sqrt{0.2121}e^{\theta_4}|A\bar{B}\bar{C}\rangle$$

Bayesian Networks – Quantum Inference

The full joint probability distribution table:

A	B	C	QPrA(A, B, C)
T	T	T	$\sqrt{0.3}e^{\theta_1} \times \sqrt{0.5}e^{\theta_2} \times \sqrt{0.8}e^{\theta_3} = 0.3464e^{\theta_1+\theta_2+\theta_3} = 0.3464e^{\theta_1}$
T	T	F	$\sqrt{0.3}e^{\theta_1} \times \sqrt{0.5}e^{\theta_2} \times \sqrt{0.2}e^{\theta_3} = 0.1732^{\theta_1+\theta_2+\theta_3} = 0.1732e^{\theta_2}$
T	F	T	$\sqrt{0.3}e^{\theta_1} \times \sqrt{0.5}e^{\theta_2} \times \sqrt{0.7}e^{\theta_3} = 0.3240^{\theta_1+\theta_2+\theta_3} = 0.3240e^{\theta_3}$
T	F	F	$\sqrt{0.3}e^{\theta_1} \times \sqrt{0.5}e^{\theta_2} \times \sqrt{0.3}e^{\theta_3} = 0.2121^{\theta_1+\theta_2+\theta_3} = 0.2121e^{\theta_4}$
F	T	T	$\sqrt{0.7}e^{\theta_7} \times \sqrt{0.5}e^{\theta_2} \times \sqrt{0.4}e^{\theta_3} = 0.3742e^{\theta_7+\theta_2+\theta_3} = 0.3742e^{\theta_5}$
F	T	F	$\sqrt{0.7}e^{\theta_7} \times \sqrt{0.5}e^{\theta_2} \times \sqrt{0.6}e^{\theta_9} = 0.4583e^{\theta_7+\theta_2+\theta_9} = 0.4583e^{\theta_6}$
F	F	T	$\sqrt{0.7}e^{\theta_7} \times \sqrt{0.5}e^{\theta_8} \times \sqrt{0.002}e^{\theta_3} = 0.0837^{\theta_7+\theta_5+\theta_8} = 0.0837e^{\theta_7}$
F	F	F	$\sqrt{0.7}e^{\theta_7} \times \sqrt{0.5}e^{\theta_8} \times \sqrt{0.98}e^{\theta_9} = 0.5857^{\theta_7+\theta_8+\theta_9} = 0.5857e^{\theta_8}$

Bayesian Networks – Quantum Inference

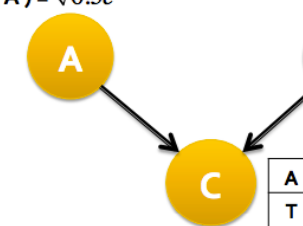
The normalized full joint probability distribution table:

A	B	C	QPrA(A, B, C)	NQPrA(A, B, C)
T	T	T	$0.3464e^{\theta_1}$	$0.6325e^{\theta_1} = \sqrt{0.4}e^{\theta_1}$
T	T	F	$0.1732e^{\theta_2}$	$0.3162e^{\theta_2} = \sqrt{0.1}e^{\theta_2}$
T	F	T	$0.3240e^{\theta_3}$	$0.5915e^{\theta_3} = \sqrt{0.35}e^{\theta_3}$
T	F	F	$0.2121e^{\theta_4}$	$0.3873e^{\theta_4} = \sqrt{0.15}e^{\theta_4}$

Bayesian Networks – Quantum Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?

Pr(A) = $\sqrt{0.3}e^{\theta 1}$ Pr(B) = $\sqrt{0.5}e^{\theta 2}$



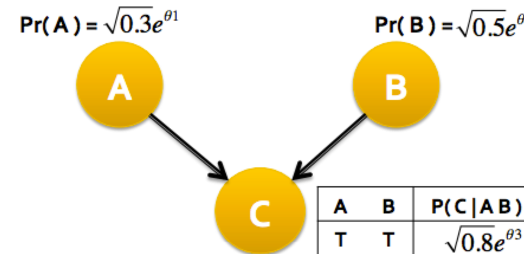
A	B	P(C | A B)
T	T	$\sqrt{0.8}e^{\theta 3}$
T	F	$\sqrt{0.7}e^{\theta 4}$
F	T	$\sqrt{0.4}e^{\theta 5}$
F	F	$\sqrt{0.02}e^{\theta 6}$

Selecting the entries of interest:

$$\left| P_{C=t|A=t,B=t}|S\rangle + P_{C=t|A=t,B=f}|S\rangle \right|^2$$

Bayesian Networks – Quantum Inference

What is the probability of node **C**, given that node **A** was **observed to occur**?



A	B	P(C | A B)
T	T	$\sqrt{0.8}e^{\theta 3}$
T	F	$\sqrt{0.7}e^{\theta 4}$
F	T	$\sqrt{0.4}e^{\theta 5}$
F	F	$\sqrt{0.02}e^{\theta 6}$

$$\left| P_{C=t|A=t,B=t}|S\rangle + P_{C=t|A=t,B=f}|S\rangle \right|^2$$

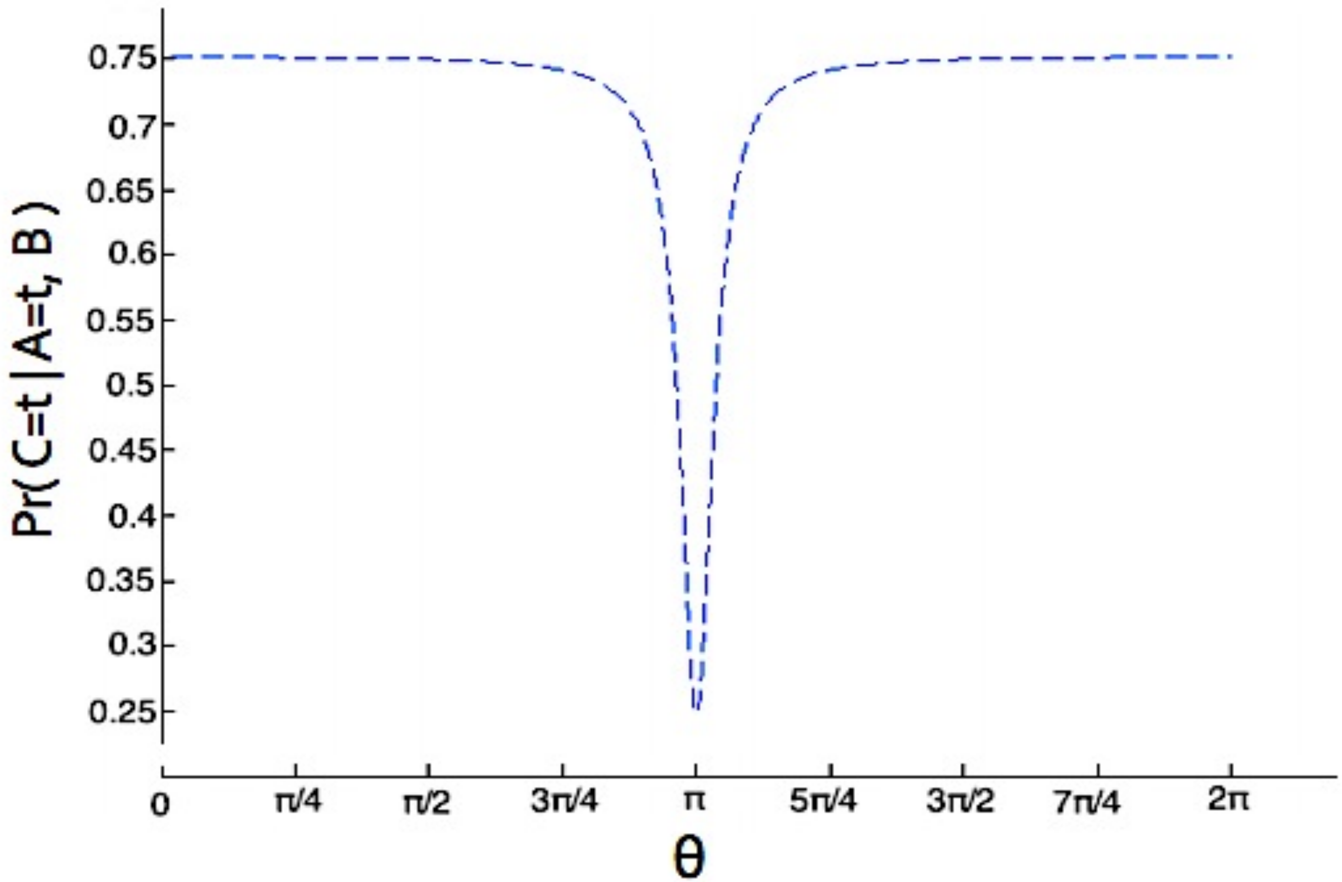
Classical Probability

$$Pr(C=t|A=t,B) = 0.75 + 2\sqrt{0.4}\sqrt{0.35}\cos(\theta_1 - \theta_2)$$

Quantum Interference

Bayesian Networks – Quantum Inference

The quantum probability can be anything!



Bayesian Networks – Quantum Inference

Problems with the current quantum Bayesian Networks of the literature:

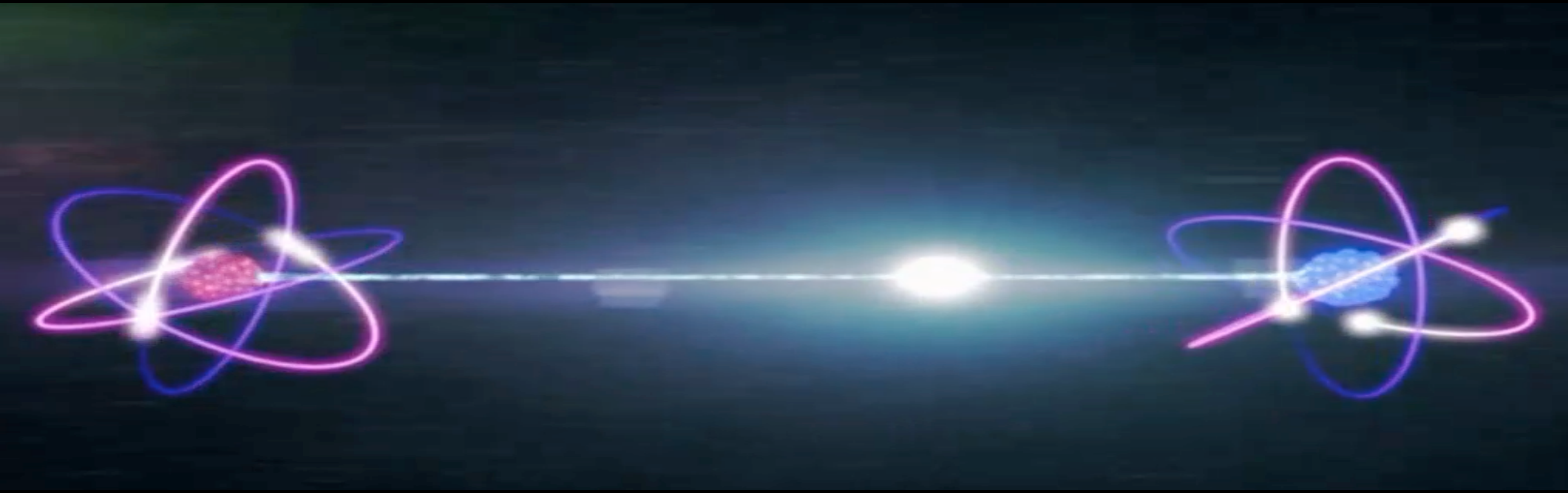
- They do not make use of **quantum interference effects** found in cognitive science literature. This means that the quantum network does not have any **advantages** compared to its classical counterpart!

Bayesian Networks – Quantum Inference

Problems with the current quantum Bayesian Networks from the literature:

- The number of **quantum parameters grow exponentially** with the amount of uncertainty in the network. There are no efforts in the literature that attempt to solve this parameter tuning automatically

Thank You!!!



Questions?